Lecture Note on Terwilliger Algebra

P. Terwilliger, edited by H. Suzuki

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- Original Hand Written Note Edited by Hiroshi Suzuki: https://icu-hsuzuki.github.io/lecturenote/
- PDF of this lecturenote: https://icu-hsuzuki.github.io/t-algebra/t-algebra.pdf
  - You can download from the download icon on the top menu.
  - The style is a bit different from the HTML version
- This digital book is created by `bookdown` package on RStudio.
  - For `bookdown` See (Xie, 2015), (Xie, 2017), (Yihui Xie, 2018).
  - See technical memo

Foreword

April 4, 1995.

This book is a lecture note based on a series of lectures by Paul Terwilliger in 1993. The original is a manuscript written by Paul Terwilliger.

This note was rewritten by Hirosh Suzuki when he studied the lecture note during the following period.

January 13, 1995 – March 4, 1995.

He had a chance to meet the author for a week after reading through the lecture note. The author clarified almost everything he asked. So even in the part where he put "?", there seems to be no mathematical gap. But sometimes, it requires lengthy calculations.

In the last part, each result has two numbers because the original lecture note has duplications. He supposes that this lecture note is already two years old, so some statements are improved essentially.

Hiroshi Suzuki

hsuzuki@icu.a.jp

Preface by P. Terwilliger

This book attempts to prepare the way for an eventual classification of the graphs that are both thin and Q-polynomial. These graphs are distance-regular or bi-distance-regular, and since the distance-regular case is somewhat easier to handle, the focus will be on that case. (It is assumed the bi-distance-regular case is not too different). In the core of this book, we take a vertex x in a distance-regular graph, and study the irreducible modules for the subconstituent algebra $T(x)$ that have endpoint at most 2. (The modules with endpoint at most 3 seems too complicated to consider, and do not seem to play much of a role anyway). The thin condition and the Q-polynomial property each affect the structure of these momdules, so these assumptions are first considered separately, and then jointly.

1. Introduction (Chapters 1 - 8)

1a. The subconstituent algebra $T(x)$ associated with any vertex x in a graph

1b. Example: The D-dimensional cube and the Lie algebra $sl_2(\mathbb{C})$

1c. The graphs of thin type: definition and characterizations

2. The structure of a thin $T(x)$-module W in an arbitrary graph (Chapters 9 - 11)

2a. The constants $a_i(W)$, $x_i(W)$

2b. The measure $m(W)$

2c. The isomorphism class of W determines and is determined by $m(W)$

2d. How non-orthogonal thin irreducible $T(x)$-modules and thin, irreducible $T(y)$-modules are related

2e. The matrices R, F, L, and R^{-1}, L^{-1}

3. Distance-regularity (Chapters 12 - 13)

3a. Distance-regularity with respect to a vertex

3b. The trivial $T(x)$ modules

3c. A graph is distance-regular with respect to each vertex if and only if the trivial $T(x)$-module is thin if and only if the graph is distance-regular or bi-distance-regular

4. The structure of a thin irreducible $T(x)$-module W with endpoint 1 in a distance-regular graph (Chapters 14 - 17)

4a. The isomorphism class of W is determined by the intersection numbers and $a_0(W)$

4b. Span($\{v_1^+, v_2^+, \ldots, v_D^+\}$) is thin irreducible $T(x)$-module if and only if v_i^+, v_i^- are dependent, for all i

Chapter 1

Subconstituent Algebra of a Graph

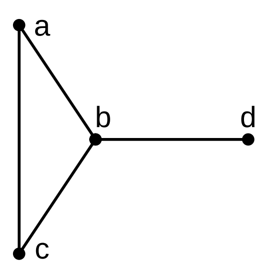
Wednesday, January 20, 1993

A graph (undirected, without loops or multiple edges) is a pair $\Gamma = (X, E)$, where

$$\begin{aligned} X &= \text{finite set (of vertices)} \\ E &= \text{set of (distinct) 2-element subsets of } X \text{ (= edges of) } \Gamma. \end{aligned}$$

The vertices x and $y \in X$ are adjacent if and only if $xy \in E$.

Example 1.1. Let Γ be a graph. $X = \{a, b, c, d\}$, $E = \{ab, ac, bc, bd\}$.



Set $n = |X|$, the order of Γ.

Pick a field K (= $\mathbb{R}$ or $\mathbb{C}$). Then $\mathrm{Mat}_X(K)$ denotes the K algebra of all $n \times n$ matrices with entries in K. (rows and columns are indexed by X)

Adjacency matrix $A \in \mathrm{Mat}_X(K)$ is defined by

$$A_{xy} = \begin{cases} 1 & \text{if } xy \in E, \\ 0 & \text{else.} \end{cases} \tag{1.1}$$

Example 1.2. Let a, b, c, d be labels of rows and columns. Then

$$A = \begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

The subalgebra M of $\mathrm{Mat}_X(K)$ generated by A is called the *Bose-Mesner algebra* of Γ.

Set $V = K^n$, the set of n-dimensional column vectors, the coordinates are indexed by X.

Let $\langle\ ,\ \rangle$ denote the Hermitean inner product:

$$\langle u, v \rangle = u^\top \cdot \bar{v} \quad (u, v \in V)$$

V with $\langle\ ,\ \rangle$ is the *standard module* of Γ.

M acts on V: For every $x \in X$, write

$$\hat{x} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow x$$

where 1 is at the x position.

Then

$$A\hat{x} = \sum_{y \in X, xy \in E} \hat{y}.$$

Since A is a real symmetrix matrix,

$$V = V_0 + V_1 + \cdots + V_r \qquad \text{some } r \in \mathbb{Z}^{\geq 0},$$

the orthogonal direct sum of maximal A-eigenspaces.

Let $E_i \in \mathrm{Mat}_X(K)$ denote the orthogonal projection,

$$E_i : V \longrightarrow V_i.$$

Then $E_0, \dots, E_r$ are the primitive idempotents of M.

$$M = \mathrm{Span}_K(E_0, \dots, E_r),$$

$$E_i E_j = \delta_{ij} E_i \quad \text{for all } \ i, j, \quad E_0 + \cdots + E_r = I.$$

Let θ_i denote the eigenvalue of A for V_i in $\mathbb{R}$. Without loss of generality we may assume that

$$\theta_0 > \theta_1 > \cdots > \theta_r.$$

Let

$$m_i = \text{the multiplicity of } \ \theta_i = \dim V_i = \mathrm{rank} E_i.$$

Set

$$\mathrm{Spec}(\Gamma) = \begin{pmatrix} \theta_0, & \theta_1, & \dots, & \theta_r \\ m_0, & m_1, & \dots, & m_r \end{pmatrix}.$$

Problem. What can we say about Γ when $\mathrm{Spec}(\Gamma)$ is given?

The following Lemma 1.1, is an example of Problem.

For every $x \in X$,

$$k(x) \equiv \text{ valency of } x \equiv \text{ degree of } x \equiv |\{y \mid y \in X,\ xy \in E\}|.$$

Definition 1.1. The graph Γ is *regular* of valency k if $k = k(x)$ for every $x \in X$.

Lemma 1.1. *With the above notation,*

(i) $\theta_0 \leq \max\{k(x) \mid x \in X\} = k^{\max}$.
(ii) If Γ *is regular of valency* k*, then* $\theta_0 = k$.

Proof. (i) Without loss of generality we may assume that $\theta_0 > 0$, else done. Let $v := \sum_{x \in X} \alpha_x \hat{x}$ denote the eivenvector for θ_0.

Pick $x \in X$ with $|\alpha_x|$ maximal. Then $|\alpha_x| \neq 0$.

Since $Av = \theta_0 v$,

$$\theta_0 \alpha_x = \sum_{y \in X, xy \in E} \alpha_y.$$

So,

$$\theta_0 |\alpha_x| = |\theta_0 \alpha_x| \leq \sum_{y \in X, xy \in E} |\alpha_y| \leq k(x) |\alpha_x| \leq k^{\max} |\alpha_x|.$$

(ii) All 1's vector $v = \sum_{x \in X} \hat{x}$ satisfies $Av = kv$.

$\square$

Let $x, y \in X$ and $\ell \in \mathbb{Z}^{\geq 0}$.

Definition 1.2. A *path* of length ℓ connecting x, y is a sequence

$$x = x_0, x_1, \dots, x_\ell = y, \quad x_i \in X \quad (0 \leq i \leq \ell)$$

such that $x_i x_{i+1} \in E$ for all i $(0 \leq i \leq \ell - 1)$.

Definition 1.3. The *distance* $\partial(x, y)$ is the length of a shortest path connecting x and y.

$$\partial(x, y) \in \mathbb{Z}^{\geq 0} \cup \{\infty\}.$$

Definition 1.4. The graph Γ is *connected* if and only if $\partial(x, y) < \infty$ for all $x, y \in X$.

From now on, assume that Γ is connected with $|X| \geq 2$.

Set

$$d_\Gamma = d = \max\{\partial(x, y) \mid x, y \in X\} \equiv \textit{the diameter of } \Gamma.$$

Definition 1.5. For each vertex $x \in X$,

$$d(x) = \textit{the diameter with respect to } x = \max\{\partial(x, y) \mid y \in X\} \leq d.$$

Fix a 'base' vertex $x \in X$.

Observe that

$$V = V_0^* + V_1^* + \cdots + V_{d(x)}^* \quad \text{(orthogonal direct sum)},$$

where

$$V_i^* = \operatorname{Span}_K(\hat{y} \mid \partial(x, y) = i) \equiv V_i^*(x)$$

and $V_i^* = V_i^*(x)$ is called the i-th subconstituent with respect to x.

Let $E_i^* = E_i^*(x)$ denote the orthogonal projection

$$E_i^* : V \longrightarrow V_i^*(x).$$

View $E_i^*(x) \in \operatorname{Mat}_X(K)$. So, $E_i^*(x)$ is diagonal with yy entry:

$$(E_i^*(x))_{yy} = \begin{cases} 1 & \text{if } \partial(x, y) = i, \\ 0 & \text{else,} \end{cases} \qquad \text{for } y \in X.$$

Set

$$M^* = M^*(x) \equiv \operatorname{Span}_K(E_0^*(x), \dots, E_{d(x)}^*(x)).$$

Then $M^*(x)$ is a commutative subalgebra of $\operatorname{Mat}_X(K)$ and is called the *dual Bose-Mesner algbara with respect to x*.

Definition 1.6 (Subconstituent Algebra)**.** Let $\Gamma = (X, E)$, x, M, $M^*(x)$ be as above. Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(K)$ generated by M and $M^*(x)$. T is the *subconstituent algebra* of Γ with respect to x.

Definition 1.7. A T-module is any subspace $W \subseteq V$ such that $aw \in W$ for all $a \in T$ and $w \in W$.

T-module W is *irreducible* if and only if $W \neq 0$ and W does not properly contain a nonzero T-module.

For any $a \in \mathrm{Mat}_X(K)$, let a^* denbote the conjugate transpose of a.

Observe that

$$\langle au, v\rangle = \langle u, a^*v\rangle \quad \text{for all} \;\; a \in \mathrm{Mat}_X(K), \;\text{and for all} \;\; u, v \in V.$$

Lemma 1.2. *Let $\Gamma = (X, E)$, $x \in X$ and $T \equiv T(x)$ be as above.*

(i) If $a \in T$, then $a^* \in T$.

(ii) For any T-module $W \subset V$,

$$W^{\perp} := \{v \in V \mid \langle w, v\rangle = 0, \; \textit{for all } w \in W\}$$

is a T-module.

(iii) V decomposes as an orthogonal direct sum of irreducible T-modules.

Proof. (i) It is because T is generated by symmetric real matrices

$$A, E_0^*(x), E_1^*(x), \dots, E_{d(x)}^*(x).$$

(ii) Pick $v \in W^{\perp}$ and $a \in T$, it suffices to show that $av \in W^{\perp}$. For all $w \in W$,

$$\langle w, av\rangle = \langle a^*w, v\rangle = 0$$

as $a^* \in T$.

(iii) This is proved by the induction on the dimension of T-modules. If W is an irreducible T-module of V, then

$$V = W + W^{\perp} \quad \text{(orthogonal direct sum)}.$$

□

Problem. What does the structure of the $T(x)$-module tell us about Γ?

Study those Γ whose modules take 'simple' form. The Γ's involved are highly regular.

HS MEMO

1. The subconstituent algebra T is semisimple as the left regular representation of T is completely reducible. See Curtis-Reiner 25.2 (Charles W. Curtis, 2006).
2. The inner product $\langle a, b\rangle_T = \mathrm{tr}(a^\top \bar{b})$ is nondegenerate on T.
3. In general,

$$
\begin{aligned}
T\text{: Semisimple and Artinian} &\Leftrightarrow T\text{: Artinian with } J(T) = 0\\
&\Leftarrow T\text{: Artinian with nonzero nilpotent element}\\
&\Leftarrow T \subset \mathrm{Mat}_X(K) \text{ such that for all } a \in T \text{ is normal.}
\end{aligned}
$$

Chapter 2

Perron-Frobenius Theorem

Friday, January 22, 1993

In this lecture, we use the Perron Frobenius theory of non-negative matrices to obtain information on eigenvalues of a graph.

Let $K = \mathbb{R}$. For $n \in \mathbb{Z}^{>0}$, pick a symmetric matrix $C \in \mathrm{Mat}_n(\mathbb{R})$.

Definition 2.1. The matrix C is *reducible* if and only if there is a bipartition $\{1, 2, \dots, n\} = X^+ \cup X^-$ (disjoint union of nonempty sets) such that $C_{ij} = 0$ for all $i \in X^+$, and for all $j \in X^-$, and for all $i \in X^-$, and for all $j \in X^+$, i.e.,

$$C \sim \begin{pmatrix} * & O \\ O & * \end{pmatrix}.$$

Definition 2.2. The matrix C is *bipartite* if and only if there is a bipartition $\{1, 2, \dots, n\} = X^+ \cup X^-$ (disjoint union of nonempty sets) such that $C_{ij} = 0$ for all $i, j \in X^+$, and for all $i, j \in X^-$, i.e.,

$$C \sim \begin{pmatrix} O & * \\ * & O \end{pmatrix}.$$

Note.

1. If C is bipartite, for every eigenvalue θ of C, $-\theta$ is an eigenvalue of C such that $\mathrm{mult}(\theta) = \mathrm{mult}(-\theta)$.

Indeed, let $C = \begin{pmatrix} O & A \\ B & O \end{pmatrix}$,

$$\begin{pmatrix} O & A \\ B & O \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \theta \begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{pmatrix} O & A \\ B & O \end{pmatrix} \begin{pmatrix} x \\ -y \end{pmatrix} = -\theta \begin{pmatrix} x \\ -y \end{pmatrix},$$

where $Ay = \theta x$ and $Bx = \theta y$.

2. If C is bipartite, C^2 is reducible.
3. The matrix C is irreducible and C^2 is reducible, if $C_{ij} \geq 0$ for all i, j and C is bipartite. (Exercise)

HS MEMO

Note 1. Even if C is not symmetric

$$\begin{pmatrix} O & A \\ B & O \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \theta \begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{pmatrix} O & A \\ B & O \end{pmatrix} \begin{pmatrix} x \\ -y \end{pmatrix} = -\theta \begin{pmatrix} x \\ -y \end{pmatrix}$$

holds. So the geometric multiplicities of θ and $-\theta$ coincide. How about the algebraic multiplicities?

Note 3. Set $x \sim y$ if and only if $C_{xy} > 0$. So the graph may have loops. Then

$$(C^2)_{xy} > 0 \Leftrightarrow \text{ if there exists } z \in X \text{ such that } x \sim z \sim y.$$

Note that C is irreducible if and only if $\Gamma(C)$ is connected. Let

$$X^+ = \{y \mid \text{there is a path of even length from } x \text{ to } y\} \tag{2.1}$$
$$X^- = \{y \mid \text{there is no path of even length from } x \text{ to } y\} \neq \emptyset. \tag{2.2}$$

If there is an edge $y \sim z$ in X^+ and $w \in X^-$. Then there would be a path from x to y of even length. So $\mathrm{e}(X^+, X^+) = \mathrm{e}(X^-, X^-) = 0$..

Theorem 2.1 (Perron-Frobenius). *Given a matrix C in $\mathrm{Mat}_n(\mathbb{R})$ such that*

(a) C is symmetric.

(b) C is irreducible.

(c) $C_{ij} \geq 0$ for all i, j.

Let θ_0 be the maximal eigenvalue of C with eigenspace $V_0 \subseteq \mathbb{R}^n$, and let θ_r be the minimal eigenvalue of C with eigenspace $V_r \subseteq \mathbb{R}^n$. Then the following hold.

(i) Suppose $0 \neq v = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \in V_0$. *Then $\alpha_i > 0$ for all i, or $\alpha_i < 0$ for all i.*

(ii) $\dim V_0 = 1$.

(iii) $\theta_r \geq -\theta_0$.

(iv) $\theta_r = \theta_0$ if and only if C is bipartite.

First, we prove the following lemma.

Lemma 2.1. *Let $\langle\ ,\ \rangle$ be the dot product in $V = \mathbb{R}^n$. Pick a symmetric matrix $B \in \mathrm{Mat}_n(\mathbb{R})$. Suppose all eigenvalues of B are nonnegative. (i.e., B is positive semidefinite.) Then there exist vectors $v_1, v_2, \dots, v_n \in V$ such that $B_{ij} = \langle v_i, v_j \rangle$ for all i, j $(1 \le i, j \le n)$.*

Proof. By elementary linear algebra, there exists an orthonormal basis $w_1, w_2, \dots, w_n$ of V consisting of eigenvectors of B. Set the i-th column of P is w_i and $D = \mathrm{diag}(\theta_1, \dots, \theta_n)$. Then $P^\top P = I$ and $BP = PD$.

Hence,

$$B = PDP^{-1} = PDP^\top = QQ^\top,$$

where

$$Q = P \cdot \mathrm{diag}(\sqrt{\theta_1}, \sqrt{\theta_2}, \dots, \sqrt{\theta_n}) \in \mathrm{Mat}_n(\mathbb{R}).$$

Now, let v_i be the i-th column of $Q^\top$. Then

$$B_{ij} = v_i^\top \cdot v_j = \langle v_i, v_j \rangle.$$

This proves the lemma. $\square$

Now we start the proof of Theorem 2.1.

Proof. (i) Let $\langle\ ,\ \rangle$ denote the dot product on $V = \mathbb{R}^n$. Set

$$\begin{aligned} B &= \theta I - C && (2.3) \\ &= \text{symmetric matrix with eigenvalues } \theta_0 - \theta_i \ge 0 && (2.4) \\ &= (\langle v_i, v_j \rangle)_{1 \le i,j \le n} && (2.5) \end{aligned}$$

with the same $v_1, \dots, v_n \in V$ by Lemma 2.1.

Observe:

$$\sum_{i=1}^n \alpha_i v_i = 0.$$

Pf.

$$\begin{aligned} \left\| \sum_{i=1}^n \alpha_i v_i \right\|^2 &= \left\langle \sum_{i=1}^n \alpha_i v_i, \sum_{i=1}^n \alpha_i v_i \right\rangle && (2.6) \\ &= (\alpha_1 \quad \dots \quad \alpha_n)\, B \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} && (2.7) \\ &= v^\top B v && (2.8) \\ &= 0, && (2.9) \end{aligned}$$

since $Bv = (\theta_0 I - C)v = 0$.

Now set

$$s = \text{the number of indices } i, \text{ where } \alpha_i > 0.$$

Replacing v by $-v$ if necessary, without loss of generality we may assume that $s \geq 1$. We want to show $s = n$.

Assume $s < n$. Without loss of generality, we may assume that $\alpha_i > 0$ for $1 \leq i \leq s$ and $\alpha_i \leq 0$ for $s+1 \leq i \leq n$. Set

$$\rho = \alpha_1 v_1 + \cdots + \alpha_s v_s = -\alpha_{s+1} v_{s+1} - \cdots - \alpha_n v_n.$$

Then, for $i = 1, \dots, s$,

$$\begin{aligned} \langle v_i, \rho \rangle &= \sum_{j=s+1}^{n} -\alpha_j \langle v_i, v_j \rangle \quad (\langle v_i, v_j \rangle = B_{ij}, B = \theta_0 I - C) && (2.10) \\ &= \sum_{j=s+1}^{n} (-\alpha_{ij})(-C_{ij}) && (2.11) \\ &\leq 0. && (2.12) \end{aligned}$$

Hence

$$0 \leq \langle \rho, \rho \rangle = \sum_{i=1}^{s} \alpha_i \langle v_i, \rho \rangle \leq 0,$$

as $\alpha_i > 0$ and $\langle v_i, \rho \rangle \leq 0$. Thus, we have $\langle \rho, \rho \rangle = 0$ and $\rho = 0$. For $j = s+1, \dots, n$,

$$0 = \langle \rho, v_j \rangle = \sum_{i=1}^{s} \alpha_i \langle v_i, v_j \rangle \leq 0,$$

as $\langle v_i, v_j \rangle = -C_{ij}$.

Therefore,

$$0 = \langle v_i, v_j \rangle = -C_{ij} \text{ for } 1 \leq i \leq s, \ s+1 \leq j \leq n.$$

Since C is symmetric,

$$C = \begin{pmatrix} * & O \\ O & * \end{pmatrix}$$

Thus C is reducible, which is not the case. Hence $s = n$.

(ii) Suppose $\dim V_0 \geq 2$. Then,

$$\dim \left(V_0 \cap \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}^{\perp} \right) \geq 1.$$

So, there is a vector

$$0 \neq v = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in V_0$$

with $\alpha_1 = 0$. This contradicts (i).

Now pick

$$0 \neq w = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} \in V_r.$$

(iii) Suppose $\theta_r < -\theta_0$. Since the eigenvalues of C^2 are the squares of those of C, θ_r^2 is the maximal eigenvalue of C^2.

Also we have $C^2 w = \theta_r^2 w$.

Observe that C^2 is irreducible. (As otherwise, C is bipartite by Note 3, and we must have $\theta_r = -\theta_0$.) Therefore, $\beta_i > 0$ for all i or $\beta_i < 0$ for all i. We have

$$\langle v, w \rangle = \sum_{i=1}^{n} \alpha_i \beta_j \neq 0.$$

This is a contradiction, as $V_0 \perp V_r$.

(iv) $\Rightarrow$: Let $\theta_r = -\theta_0$. Then $\theta = \theta_1^2 = \theta_0^2$ is the maximal eigenvalue of C^2, and v and w are linearly independent eigenvalues for θ for C^2. Hence, for C^2, $\mathrm{mult}(\theta) \geq 2$.

Thus by (ii), C^2 must be reducible. Therefore, C is bipartite by Note 3.

$\Leftarrow$: This is Note 1. □

Let $\Gamma = (X, E)$ be any graph.

Definition 2.3. Γ is said to be *bipartite* if the adjacency matrix A is bipartite. That is, X can be written as a disjoint union of X^+ and X^- such that X^+, X^- contain no edges of Γ.

Corollary 2.1. *For any (connected) graph* Γ *with*

$$\mathrm{Spec}(\Gamma) = \begin{pmatrix} \theta_0 & \theta_1 & \cdots & \theta_r \\ m_1 & m_1 & \cdots & m_r \end{pmatrix} \quad \textit{with } \theta_0 > \theta_1 > \cdots > \theta_r.$$

Let V_i *be the eigenspace of* θ_i*. Then the following holds.*

1. *Supppose* $0 \neq v = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in V_0 \in \mathbb{R}^n$*. Then* $\alpha_i > 0$ *for all* i*, or* $\alpha_i < 0$ *for all* i*.*
2. $m_0 = 1$*.*

3. *$\theta_r \geq -\theta_0$ if and only if Γ is bipartite. In this case,*

$$-\theta_i = \theta_{r-i} \text{ and } m_i = m_{r-i} \quad (0 \leq i \leq r).$$

Proof. This is a direct consequences of Theorem 2.1 and Note 3. $\square$

Chapter 3

Cayley Graphs

Monday, January 25, 1993

Given graphs $\Gamma = (X, E)$ and $\Gamma' = (X', E')$.

Definition 3.1. A map $\sigma : X \to X'$ is an *isomorphism* of graphs whenever;

i. σ is one-to-one and onto,
ii. $xy \in E$ if and only if $\sigma x \sigma y \in E'$ for all $x, y \in X$.

We do not distinguish between isomorphic graphs.

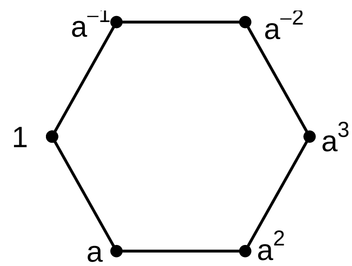
Definition 3.2. Suppose $\Gamma = \Gamma'$. Above isomorphism σ is called an *automorphism* of Γ. Then set $\mathrm{Aut}(\Gamma)$ of all automorphisms of Γ becomes a finite group under composition.

Definition 3.3. If $\mathrm{Aut}(\Gamma)$ acts transitive on X, Γ is called *vertex transitive*.

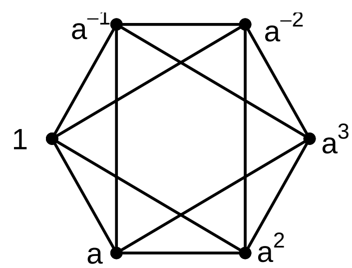
Definition 3.4 (Cayley Graphs)**.** Let G be any finite group, and Δ any generating set for G such that $1_G \notin \Delta$ and $g \in \Delta \to g^{-1} \in \Delta$. Then Cayley graph $\Gamma = \Gamma(G, \Delta)$ is defined on the vetex set $X = G$ with the edge set E define by the following.

$$E = \{(h_1, h_2) \mid h_1, h_2 \in G, h_1^{-1} h_2 \in \Delta\} = \{(h, hg) \mid h \in G, g \in \Delta\}.$$

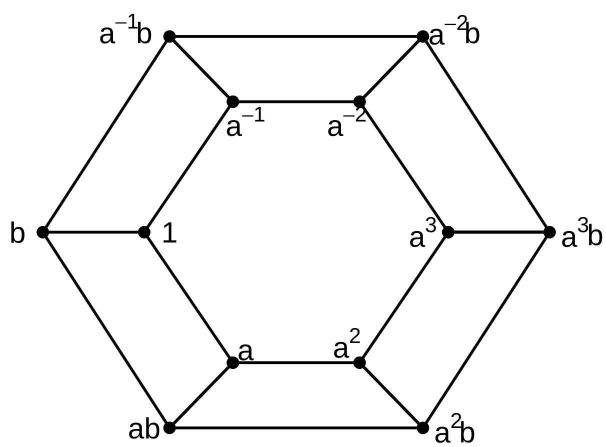
Example 3.1. $G = \langle a \mid a^6 = 1 \rangle$, $\Delta = \{a, a^{-1}\}$.



Example 3.2. $G = \langle a \mid a^6 = 1 \rangle$, $\Delta = \{a, a^{-1}, a^2, a^{-2}\}$.



Example 3.3. $G = \langle a, b \mid a^6 = 1 = b^2, ab = ba \rangle$, $\Delta = \{a, a^{-1}, b\}$.



HS MEMO

$\mathrm{Aut}(\Gamma) \simeq D_6 \times \mathbb{Z}_2$ contains two regular subgroups isomorphic to D_6 and $\mathbb{Z}_6 \times \mathbb{Z}_2$ and Γ is obtained as Cayley graphs in two ways.

Cayley graphs are vertex transitive, indeed.

Theorem 3.1. *The following hold.*

(*i*) *For any Cayley graph* $\Gamma = \Gamma(G, \Delta)$, *the map*

$$G \to \mathrm{Aut}(\Gamma)\ (g \mapsto \hat{g})$$

is an injective homomorphism of groups, where

$$\hat{g}(x) = gx \quad \textit{for all } g \in G \textit{ and for all } x \in X(= G).$$

Also, the image $\hat{G}$ *is regular on* X*. i.e., the image* $\hat{G}$ *acts transitively on* X *with trivial vertex stabilizers.*

(ii) For any graph $\Gamma = (X, E)$*, suppose there exists a subgroup* $G \subseteq \mathrm{Aut}(\Gamma)$ *that is regular on* X*. Pick* $x \in X$*, and let*

$$\Delta = \{g \in G \mid \langle x, g(x) \in E\}.$$

Then $1 \notin \Delta$*,* $g \in \Delta \to g^{-1} \in \Delta$*, and* Δ *generates* G*. Moreover,* $\Gamma \simeq \Gamma(G, \Delta)$*.*

Proof. (i) Let $g \in G$. We want to show that $\hat{g} \in \mathrm{Aut}(\Gamma)$. Let $h_1, h_2 \in X = G$. Then,

$$\begin{aligned}
(h_1, h_2) \in E &\to h_1^{-1} h_2 \in \Delta && (3.1)\\
&\to (gh_1)^{-1}(gh_2) \in \Delta && (3.2)\\
&\to (gh_1, gh_2) \in E && (3.3)\\
&\to (\hat{g}(h_1), \hat{g}(h_2)) \in E. && (3.4)
\end{aligned}$$

Hence, $\hat{g} \in \mathrm{Aut}(\Gamma)$.

Observe: $g \mapsto \hat{g}$ is a homomorphism of groups:

$$\hat{1}_G = 1,\ \widehat{g_1 g_2} = \widehat{g_1}\widehat{g_2}.$$

Observe: $g \mapsto \hat{g}$ is one-to-one:

$$\widehat{g_1} = \widehat{g_2} \to g_1 = \widehat{g_1}(1_G) = \widehat{g_2}(1_G) = g_2.$$

Observe: $\hat{G}$ is regular on X: Clear by construction.

(ii) $1_G \notin \Delta$: Since Γ has not loops, $(x, 1_G x) \notin E$.

$g \in \Delta \to g^{-1} \in \Delta$:

$$g \in \Delta \to (x, g(x)) \in E \to E \ni (g^{-1}(x), g^{-1}(g(x))) = (g^{-1}(x), x).$$

Δ generates G: Suppose $\langle \Delta \rangle \subsetneq G$. Let $\hat{X} = \{g(x) \mid g \in \langle \Delta \rangle\} \subsetneq X$. ($\hat{X} \subsetneq X$ as G acts regularly on X.)

Since Γ is connected, there exists $y \in \hat{X}$ and $z \in X \quad \hat{X}$ with $yz \in E$.

Let $y = g(x)$, $g \in \langle \Delta \rangle$, $z \in h(x)$, $h \in G \quad \langle \Delta \rangle$. Then

$$(y, z) = (g(x), h(x)) \in E \to (x, g^{-1}h(x)) \in E \to g^{-1}h \in \langle \Delta \rangle \to h \in \langle \Delta \rangle.$$

This is a contradition. Therefore, Δ generates G.

Let $\Gamma' = (X', E')$ denote $\Gamma(G, \Delta)$. We shall show that

$$\theta : X' \to X \ (g \mapsto g(x))$$

is an isomorphism of graphs.

θ is one-to-one: For $h_1, h_2 \in X' = G$,

$\theta(h_1) = \theta(h_2) \to h_1(x) = h_2(x) \to h_2^{-1}h_1(x) = x \to h_2^{-1}h_1 \in \mathrm{Stab}_G(x) = \{1_G\} \to h_1 = h_2.$

$(\mathrm{Stab}_G(x) = \{g \in G \mid g(x) = x\}.)$

θ is onto: Since G is transitive,

$$X = \{g(x) \mid g \in G\} = \theta(X') = \theta(G).$$

θ respects adjacency: For $h_1, h_2 \in X' = G$,

$(h_1, h_2) \in E' \leftrightarrow h_1^{-1}h_2 \in \Delta \leftrightarrow (x, h_1^{-1}h_2(x)) \in E \leftrightarrow (h_1(x), h_2(x)) \in E \leftrightarrow (\theta(h_1), \theta(h_2)) \in E.$

Therefore θ is an isomorphism between graphs $\Gamma(G, \Delta)$ and $\Gamma(X, E)$. $\square$

How to compute the eigenvalues of the Cayley graph of and abelian group.

Let G be any finite abelian group. Let $\mathbb{C}^*$ be the multiplicative group on $\mathbb{C}\ \{0\}$.

Definition 3.5. A (linear) G-character is any group homomorphism $\theta : G \to \mathbb{C}^*$.

Example 3.4. $G = \langle a \mid a^3 = 1\rangle$ has three characters, $\theta_0, \theta_1, \theta_2$.

$$\begin{array}{c|ccc} \theta_i(a^j) & 1 & a & a^2 \\ \hline \theta_0 & 1 & 1 & 1 \\ \theta_1 & 1 & \omega & \omega^2 \\ \theta_2 & 1 & \omega^2 & \omega \end{array}, \quad \text{with} \ \ \omega = \frac{-1+\sqrt{-3}}{2}.$$

Here ω is a primitive cube root of q in $\mathbb{C}^*$, i.e., $1 + \omega + \omega^2 = 0$.

For arbitraty group G, let $X(G)$ be the set of all characters of G.

Observe: For $\theta_1, \theta_2 \in X(G)$, one can define product $\theta_1\theta_2$:

$$\theta_1\theta_2(g) = \theta_1(g)\theta_2(g) \quad \text{for all} \ \ g \in G.$$

Then $\theta_1\theta_2 \in X(G)$.

Observe: $X(G)$ with this product is an (abelian) group.

Lemma 3.1. *The groups G and $X(G)$ are isomorphic for all finite abelian groups G.*

Proof. G is a direct sum of cyclic groups;

$$G = G_1 \oplus G_2 \oplus \cdots \oplus G_m, \quad \text{where} \;\; G_i = \langle a_i \mid a_i^{d_i} = 1 \rangle \quad (1 \le i \le m).$$

Pick any element ω_i of order d_i in $\mathbb{C}^*$, i.e., a primitive d_i-the root of 1. Define

$$\theta_i : G \to \mathbb{C}^* \quad (a_1^{\varepsilon_1} \cdots a_m^{\varepsilon_m} \mapsto \omega_i^{\varepsilon_i} \quad \text{where} \;\; 0 \le \varepsilon_i < d_i, 1 \le i \le m).$$

Then $\theta_i \in X(G)$. (Exercise)

Claim: There exists an isomorphism of groups $G \to X(G)$ that sends a_i to θ_i.

Observe: $\theta_i^{d_i} = 1$. For every $g = a_1^{\varepsilon_1} \cdots a_m^{\varepsilon_m} \in G$,

$$\theta_i^{d_i}(g) = (\theta_i(g))^{d_i} = (\omega_i^{\varepsilon_i})^{d_i} = (\omega_i^{d_i})^{\varepsilon_i} = 1.$$

Observe: If $\theta_1^{\varepsilon_1}\theta_2^{\varepsilon_2} \cdots \theta_m^{\varepsilon_m} = 1$ for some $0 \le \varepsilon_i < d_i, 1 \le i \le m$. Then $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_m = 0$.

Pf. $1 = \theta_1^{\varepsilon_1}\theta_2^{\varepsilon_2} \cdots \theta_m^{\varepsilon_m}(a_i) = \omega_i^{\varepsilon_i}$, Since ω_i is a primitive d_i-th root of 1, $\varepsilon_i = 0$ for $1 \le i \le m$.

Observe: $\theta_1, \dots, \theta_m$ generate $X(G)$. Pick $\theta \in X(G)$. Since $a_i^{d_i} = 1$, $1 = \theta(a_i^{d_i}) = \theta(a_i)^{d_i}$.

Hence $\theta(a_i) = \omega^{\varepsilon_i}$ for some ε_i with $0 \le \varepsilon_i < d_i$.

Now $\theta = \theta_1^{\varepsilon_1} \cdots \theta_m^{\varepsilon_m}$, since these are both equal to $\omega_i^{\varepsilon_i}$ at a_i for $1 \le i \le m$.

Therefore,

$$G \to X(G) \quad (a_i \mapsto \theta_i)$$

is an isomorphism of groups. □

Note. The correspondence above is clearly a group homomorphism.

Chapter 4

Examples

Wednesday, January 27, 1993

Theorem 4.1. *Given a Cayley graph* $\Gamma = \Gamma(G, \Delta)$. *View the standard module* $V \equiv \mathbb{C}G$ *(the group algebra), so*

$$\left\langle \sum_{g \in G} \alpha_g g, \ \sum_{g \in G} \beta_g g \right\rangle = \sum_{g \in G} \alpha_g \overline{\beta_g}, \quad \text{with } \alpha_g, \beta_g \in \mathbb{C}.$$

For any $\theta \in X(G)$, *write*

$$\hat{\theta} = \sum_{g \in G} \theta(g^{-1}) g.$$

Then the following hold.

(i) $\langle \hat{\theta}_1, \hat{\theta}_2 \rangle = |G|$ *if* $\theta_1 = \theta_2$ *and* 0 *othewise for* $\theta_1, \theta_2 \in X(G)$. *In particular,* $\{\hat{\theta} \mid \theta \in X(G)\}$ *forms a basis for* V.

(ii) $A\hat{\theta} = \Delta_\theta \hat{\theta}$ *for* $\theta \in X(G)$, *where* A *is the adjacency matrix and*

$$\Delta_\theta = \sum_{g \in \Delta} \theta(g).$$

In particular, the eigenvalues of Γ *are precisely*

$$\{\Delta_\theta \mid \theta \in X(G)\}.$$

Proof.

(i) Claim: For every $\theta \in X(G)$, let

$$s := \sum_{g \in G} \theta(g^{-1}) = \begin{cases} |G| & \text{if } \ \theta = 1 \\ 0 & \text{if } \ \theta \neq 1. \end{cases}$$

Pf. Clear if $\theta = 1$.

Let $\theta \neq 1$. Then $\theta(h) \neq 1$ for some $h \in G$.

$$s \cdot \theta(h) = \left(\sum_{g \in G} \theta(g^{-1}) \right) \theta(h) = \sum_{g \in G} \theta(g^{-1}h) = \sum_{g' \in G} \theta(g'^{-1}) = s.$$

Since $\theta(h) \neq 1$, $s = 0$.

Claim. $\theta(g^{-1}) = \overline{\theta(g)}$ for every $\theta \in X(G)$ and every $g \in G$.

Since $\theta(g) \in \mathbb{C}$ is a root of 1,

$$|\theta(g)|^2 = \theta(g)\overline{\theta(g)} = 1.$$

On the other hand, since θ is a homomorphism,

$$\theta(g)\theta(g^{-1}) = \theta(1) = 1.$$

Hence $\theta(g^1) = \overline{\theta(g)}$.

Now

$$\begin{aligned}
\langle \widehat{\theta_1}, \widehat{\theta_2} \rangle &= \sum_{g \in G} \theta_1(g^{-1}) \overline{\theta_2(g^{-1})} && (4.1) \\
&= \sum_{g \in G} \theta_1(g^{-1}) \theta_2(g) && (4.2) \\
&= \sum_{g \in G} \theta_1 \theta_2^{-1}(g^{-1}) && (4.3) \\
&= \begin{cases} |G| & \text{if} \quad \theta_1\theta_2^{-1} = 1 \\ 0 & \text{if} \quad \theta_1\theta_2^{-1} \neq 1. \end{cases} && (4.4)
\end{aligned}$$

Since $|G| = |X(G)|$ by Lemma 3.1, and $\widehat{\theta_i}$'s are orthogonal nonzero elements in V, that form a basis of V.

(ii) Let $\Delta = \{g_1, \dots, g_r\}$. Then

$$
\begin{aligned}
A\hat{\theta} &= A\left(\sum_{g\in G}\theta(g^{-1}g)\right) && (4.5)\\
&= \sum_{g\in G}\theta(g^{-1})(gg_1+\cdots+gg_r) \quad (\Gamma(g)=\{gg_1,\ldots,gg_r\}) && (4.6)\\
&= \sum_{i=1}^{r}\left(\sum_{g\in G}\theta(g^{-1})(gg_i)\right) && (4.7)\\
&= \sum_{i=1}^{r}\left(\sum_{g\in G}\theta(g_ig_i^{-1}g^{-1})(gg_i)\right) && (4.8)\\
&= \sum_{i=1}^{r}\left(\sum_{g\in G}\theta(g_i)\theta((gg_i)^{-1})gg_i\right) && (4.9)\\
&= \sum_{i=1}^{r}\theta(g_i)\sum_{h\in G}\theta(h^{-1})h && (4.10)\\
&= \Delta_\theta\cdot\hat{\theta}. && (4.11)
\end{aligned}
$$

Since $\{\hat{\theta}\mid\theta\in X(G)\}$ forms a basis, the eigenvalues of Γ are precisely,

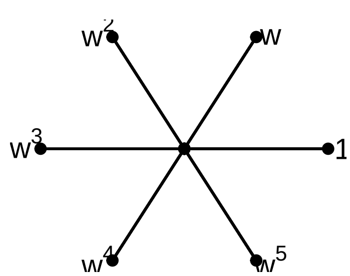
$$\{\Delta_\theta\mid\theta\in X(G)\}.$$

This completes the proof.

□

Example 4.1. Let $G=\langle a\mid a^6=1\rangle$, and $\Delta=\{a,a^{-1}\}$. Pick a primitive 6-th root of 1, ω. Then

$$X(G)=\{\theta^i\mid 0\le i\le 5\}\quad\text{such that}\quad \theta(a)=\omega,\ \omega+\omega^{-1}=1.$$



$\varphi \in X(G)$	$\varphi(a)$	$\Delta_\varphi = \theta(a) + \theta(a)^{-1}$
1	1	2
θ	ω	$\omega + \omega^{-1} = 1$
θ^2	ω^2	-1
θ^3	$\omega^3 = -1$	-2
θ^4	ω^4	-1
θ^5	ω^5	1

$$\mathrm{Spec}(\Gamma) = \begin{pmatrix} 2 & 1 & -1 & -2 \\ 1 & 2 & 2 & 1 \end{pmatrix}.$$

Example 4.2. D-cube, $H(D,2)$. Let

$$X = \{(a_1, \dots, a_D) \mid a_i \in \{1, -1\},\ 1 \le i \le D\},$$

$$E = \{xy \mid x, y \in X,\ x, y\text{: different in exactly one coordinate}\}.$$

Also $H(D,2)$ is a Cayley graph $\Gamma(G, \Delta)$, where

$$G = G_1 \oplus G_2 \oplus \cdots \oplus G_D,$$

$$G_i = \langle a_i \mid a_i^2 = 1\rangle, \quad \Delta = \{a_1, \dots, a_D\}.$$

Homework: The spectrum of $H(D,2)$ is

$$\begin{pmatrix} \theta_0 & \theta_1 & \cdots & \theta_D \\ m_0 & m_1 & \cdots & m_D \end{pmatrix},$$

where

$$\theta_i = D - 2i \quad (0 \le i \le D), \quad m_i = \binom{D}{i}.$$

HS MEMO

Let $\theta \in X(G)$. Then $\theta : X \to \{\pm 1\}$. If

$$\nu(\theta) = |\{i \mid \theta(a_i) = -1\}|,$$

then $\Delta_\theta = D - 2i$. Since there are $\binom{D}{i}$ such θ, we have te assertion.

We want to compute the subconstituent algebra for $H(D,2)$. First, we make a few observations about arbitrary graphs.

Let $\Gamma = (X, E)$ be any graph, A, the adjacemcy matrix of Γ, and V, the standard module over $K = \mathbb{C}$.

Fix a base $x \in X$. Write $E_i^* = E_i^*(x)$, and

$$T \equiv T(x) = \text{the algebra generated by } A, E_0^*, E_1^*, \dots.$$

Definition 4.1. Let W be any irreducible T-module ($\subseteq V$). Then the endpoint $r \equiv r(W)$ satisfied

$$r = \min\{i \mid E_i^* W \neq 0\}.$$

The diameter $d = d(W)$ satisfied

$$d = |\{i \mid E_i^* W \neq 0\}| - 1.$$

Lemma 4.1. *With the above notation, let W be an irreducible T-module. Then*

(i) $E_i^* A E_j^* = 0$ *if* $|i-j| > 1$, $E_i^* A E_j^* \neq 0$ *if* $|i-j| = 1$, $0 \leq i, j \leq d(x)$.
(ii) $AE_j^* W \subseteq E_{j-1}^* W + E_j^* W + E_{j+1}^* W$, $0 \leq j \leq d(x)$. *(*$E_i^* W = 0$ *if* $i < j$ *or* $i > d(x)$.*)*
(iii) $E_j^* W \neq 0$ *if* $r \leq j \leq r+d$, $= 0$ *if* $0 \leq j \leq r$ *or* $r + d < j \leq d(x)$.
(iv) $E_i^* A E_j^* W \neq 0$, *if* $|i-j| = 1$ $(r \leq i, j \leq r+d)$.

Proof.

(i) Pick $y \in X$ with $\partial(x,y) = j$. We want to find $E_i^* A E_j^* \hat{y}$. Note,

$$E_j^* \hat{y} = \begin{cases} 0 & \text{if } \partial(x.y) \neq j \\ \hat{y} & \text{if } \partial(x,y) = j. \end{cases}.$$

$$\begin{aligned} E_i^* A E_j^* \hat{y} &= E_i^* A \hat{y} && (4.12) \\ &= E_i^* \sum_{z \in X, yz \in E} \hat{z} && (4.13) \\ &= \sum_{z \in X, yz \in E, \partial(x,z) = i} \hat{z} && (4.14) \\ &= 0 \text{ if } |i-j| > 1 \qquad \text{by triangle inequality.} && (4.15) \end{aligned}$$

If $|i-j| = 1$, there exist $y, y' \in X$ such that $\partial(x,y) = j$, $\partial(x,y') = i$, $yy' \in E$ by connectivity of Γ. Hence (4.14) contains $\widehat{y'}$ and (4.14) is not equal to zero.

(ii) We have

$$\begin{aligned} AE_j^* W &= \left(\sum_{i=0}^{d(x)} E_i^* \right) AE_j^* W && (4.16) \\ &= E_{j-1}^* A E_j^* W + E_j^* A E_j^* W + E_{j+1}^* A E_j^* W && (4.17) \\ &\subseteq E_{j-1}^* W + E_j^* W + E_{j+1}^* W. && (4.18) \end{aligned}$$

(iii) Suppose $E_j^* W = 0$ for some j $(r \leq j \leq r+d)$. Then $r < j$ by the definition of r. Set

$$\widetilde{W} = E_r^* W + E_{r+1}^* W + \cdots + E_{j-1}^* W.$$

Observe $0 \subsetneq \widetilde{W} \subsetneq W$. Also $A\widetilde{W} \subseteq \widetilde{W}$ by (ii), and $E_i^* \widetilde{W} \subseteq \widetilde{W}$ for every i by construction.

Thus, $T\widetilde{W} \subseteq \widetilde{W}$, contradicting W being irreducible.

□

Chapter 5

T-Modules of $H(D,2)$, I

Friday, January 29, 1993

Let $\Gamma = (X, E)$ be a graph, A the adjacency matrix, and V the standard module over $K = \mathbb{C}$.

Fix a base $x \in X$ and write $E_i^* \equiv E_i^*(x)$, and $T \equiv T(x)$.

Let W be an irreducible T-module with endpoint $r := \min\{i \mid E_i^* W \neq 0\}$ and diameter $d := |\{i \mid E_i^* W \neq 0\}| - 1$.

We have

$$\begin{aligned} E_i^* W &\neq 0 & r \leq i \leq r+d \qquad (5.1)\\ &= 0 & 0 \leq i < r \ \text{ or } \ r+d < i \leq d(x). \qquad (5.2) \end{aligned}$$

Claim: $E_i^* A E_j^* W \neq 0$ if $|i-j| = 1$ for $r \leq i, j \leq r+d$. (See Lemma 4.1.)

Suppose $E_{j+1}^* A E_j^* W = 0$ for some j with $r \leq j < r+d$. Observe that

$$\tilde{W} = E_r^* W + \cdots + E_j^* W$$

is T-invariant with

$$0 \subsetneq \tilde{W} \subsetneq W.$$

Becase $A\tilde{W} \subseteq \tilde{W}$ since $AE_j^* W \subseteq E_{j-1}^* W + E_j^* W$,

$$E_k^* \tilde{W} \subseteq \tilde{W} \quad \text{for all} \ \ k,$$

we have $T\tilde{W} \subseteq W$.

Suppose $E_{i-1}^* A E_i^* W = 0$ for some i with $r \leq i < r+d$.

Similarly,

$$\tilde{W} = E_i^* W + \cdots + E_{r+d}^* W$$

is a T-module with $0 \subsetneq \tilde{W} \subsetneq W$.

Definition 5.1. Let Γ, E_i^*, and T be as above. Irreducible T-modules W and W' are isomorphic whenever there is an isomorphism $\sigma : W \to W'$ of vector spaces such that $a\sigma = \sigma a$ for all $a \in T$.

Recall that the standard module V is an orthogonal direct sum of irreducible T-modules
$$W_1 \oplus W_2 \oplus \cdots \oplus W_\ell, \text{ for some } \ell.$$
Given W in this list, the multiplicity of W in V is
$$|\{j \mid W_j \simeq W\}|.$$

HS MEMO

It is known that the multiplicity does not depend on the decomposition.

Now assume that Γ is the D-cube, $H(D,2)$ with $D \geq 1$. View
$$X = \{a_1 \cdots a_D \mid a_i \in \{1,-1\}, 1 \leq i \leq D\}, \tag{5.3}$$
$$E = \{xy \mid x,y \in X,\ x,y \text{ differ in exactly 1 coordinate}\}. \tag{5.4}$$

Find T-modules.

Claim: $H(D,2)$ is bipartite with a partition $X = X^+ \cup X^-$, where
$$X^+ = \{a_1 \cdots a_D \in X \mid \prod a_i > 0\} \tag{5.5}$$
$$X^- = \{a_1 \cdots a_D \in X \mid \prod a_i < 0\} \tag{5.6}$$

Observe: for all $y, z \in X$,
$$\partial(y,z) = i \Leftrightarrow y,z \text{ differ in exactly in } i \text{ coorinates with } 0 \leq i \leq D.$$

Here, the diameter of $H(D,2) = D = d$ for all $x \in X$.

Theorem 5.1. *Let $\Gamma = H(D,2)$ be as above. Fix $x \in X$, and write $E_i^* = E_i^*(x)$, and $T = T(x)$.*

Let W be an irreducible T-module with endpoint r, and diameter d with $0 \leq r \leq r + d \leq D$.

(i) W has a basis $w_0, w_1, \ldots, w_d$ with $w_i \in E_{i+r}^* W$ for $0 \leq i \leq d$. With respect to which the matrix representing A is

$$\begin{pmatrix}
0 & d & 0 & \cdots & 0 & 0 & 0 \\
1 & 0 & d-1 & \cdots & 0 & 0 & 0 \\
0 & 2 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 3 & \cdots & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \ddots & \ddots & \cdots & \cdots \\
0 & 0 & 0 & \ddots & 0 & 2 & 0 \\
0 & 0 & 0 & \cdots & d-1 & 0 & 1 \\
0 & 0 & 0 & \cdots & 0 & d & 0
\end{pmatrix}$$

(ii) $d = D - 2r$. *In particular,* $0 \leq r \leq D/2$.

(iii) Let W' *denote an irreducible* T*-module with endpoint* r'. *Then* W *and* W' *are isormorphic as* T*-modules if and only if* $r = r'$.

(iv) The multiplicity of the irreducible T*-module with endpoint* r *is*

$$\binom{D}{r} - \binom{D}{r-1} \quad \textit{if } 1 \leq r \leq R/2,$$

and 1 *if* $r = 0$.

Proof. Recall that Γ is vertex transitive. It is a Cayley graph.

Hence without loss of generality, we may assume that $x = \overbrace{11 \cdots 1}^{D}$.

Notation: Set $\Omega = \{1, 2, \ldots, D\}$. For every subset $S \subseteq \Omega$, let

$$\hat{S} = a_1 \cdots a_d \in X \quad a_i = \begin{cases} -1 & \text{if } \; i \in S \\ 1 & \text{if } i \notin S. \end{cases}$$

In particular, $\hat{\emptyset} = x$ and

$$|S| = i \Leftrightarrow \partial(x, \hat{S}) = i \Leftrightarrow \hat{S} \in E_i^* V.$$

For all $S, T \subseteq \Omega$, we say S covers T if and only if $S \supseteq T$ and $|S| = |T| + 1$.

Observe that $\hat{S}, \hat{T}$ are adjacent in Γ if and only if either T coverse S or S coverr T.

Define the 'raising matrix'

$$R = \sum_{i=0}^{D} E_{i+1}^* A E_i^*.$$

Observe that

$$RE_i^* V \subseteq E_{i+1}^* V \;\text{ for }\; 0 \leq i \leq D, \;\text{ and } E_{D+1}^* V = 0.$$

Indeed for any $S \subseteq \Omega$ with $|S| = i$,

$$\begin{aligned} R\hat{S} &= RE_i^* \hat{S} && (5.7) \\ &= E_{i+1}^* A \hat{S} && (5.8) \\ &= \sum_{T_1 \subseteq \Omega, S \text{ covers } T_1} E_{i+1}^* \widehat{T_1} + \sum_{T \subseteq \Omega, T \text{ covers } S} E_{i+1}^* \hat{T} && (5.9) \\ &= \sum_{T \subseteq \Omega, T \text{ covers } S} E_{i+1}^* \hat{T}. && (5.10) \end{aligned}$$

Define the 'lowering matrix'

$$L = \sum_{i=0}^{D} E^*_{i-1} A E^*_i.$$

Observe that

$$LE^*_i V \subseteq E^*_{i-1}V \ \text{ for } \ 0 \le i \le D, \ \text{ and } E^*_{-1}V = 0.$$

Indeed for any $S \subseteq \Omega$,

$$L\hat{S} = \sum_{T \subseteq \Omega, S \text{ covers } T} \hat{T}.$$

Observe that $A = L + R$.

For convenience, set

$$A^* = \sum_{i=0}^{D} (D - 2i) E^*_i.$$

Claim: The following hold.

(a) $LR - RL = A^*$.
(b) $A^*L - LA^* = 2L$.
(c) $A^*R - RA^* = -2R$.

In particular Span(R, L, A^*) is a 'representation of Lie algebra $\mathrm{sl}_2(\mathbb{C})$.

HS MEMO

$$\mathrm{sl}_2(\mathbb{C}) = \{X \mid \mathrm{Mat}(\mathbb{C} \mid \mathrm{tr}(X) = 0\}.$$

For $X, Y \in \mathrm{sl}_2(\mathbb{C})$, define a binary operation $[X, Y] = XY - YX$.

$$A^* \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad L \sim \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad R \sim \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then these satisfy the relations (a) - (c) above.

Proof of Claim. Apply both sides to $\hat{S}$ $\quad (S \subseteq \Omega)$. Say $|S| = i$.

Proof of (a):

$$
\begin{aligned}
(LR - RL)\hat{S} &= L\left(\sum_{\substack{T\subseteq\Omega, T \text{ covers } S\\ (D-i \text{ of them})}} \hat{T}\right) - R\left(\sum_{\substack{U\subseteq\Omega, S \text{ covers } U\\ (i \text{ of them})}} \hat{U}\right) &(5.11)\\
&= (D-i)\hat{S} + \sum_{V\subseteq\Omega, |V|=i, |S\cap V|=i-1} \hat{V} &(5.12)\\
&\quad - \left(i\hat{S} + \sum_{V\subseteq\Omega, |V|=i, |S\cap V|=i-1} \hat{V}\right) &(5.13)\\
&= (D-2i)\hat{S} &(5.14)\\
&= A^*\hat{S}. &(5.15)
\end{aligned}
$$

Proof of (b):

$$
\begin{aligned}
(A^*L - LA^*)\hat{S} &= (D-2(i-1))L\hat{S} - (D-2i)L\hat{S} \quad (\text{since } L\hat{S} \in E^*_{i-1}V) &(5.16)\\
&= 2L\hat{S}. &(5.17)
\end{aligned}
$$

Proof of (c):

$$
\begin{aligned}
(A^*R - RA^*)\hat{S} &= (D-2(i+1))R\hat{S} - (D-2i)R\hat{S} \quad (\text{since } R\hat{S} \in E^*_{i+1}V) &(5.18)\\
&= -2R\hat{S}. &(5.19)
\end{aligned}
$$

Let W be an irreducible T-module with endpoint r and diameter d ($0 \le r \le r + d \le D$).

Proof of (i) and (ii):

Pick $0 \neq w \in E^*_r W$.

Claim: $LRw = (D - 2r)w$.

Pf.

$$
\begin{aligned}
LRw &= (A^* + RL)w \quad (\text{by Claim } (a)) &(5.20)\\
&= A^*w \quad (Lw \in E^*_{r-1}W = 0) &(5.21)\\
&= (D-2r)w. &(5.22)
\end{aligned}
$$

Define

$$
w_i = \frac{1}{i!}R^i w \in E^*_{r+i}W \quad (0 \le i \le d).
$$

Then,

$$
\begin{aligned}
Rw_i &= (i+1)w_{i+1} \quad (0 \le i \le d) &(5.23)\\
Rw_d &= 0 \quad (\text{by definition of } d) &(5.24)
\end{aligned}
$$

Claim: $Lw_0 = 0$ and

$$Lw_i = (D-2r-i+1)w_{i-1} \quad (1 \le i \le d).$$

Pf. We prove by induction on i. The case $i=0$ is trivial, and the case $i=1$ follows from above claim. Let $i \ge 2$,

$$Lw_i = \frac{1}{i}LRw_{i-1} = \frac{1}{i}(A^* + RL)w_{i-1} \quad \text{(by Claim (a))} \tag{5.25}$$

$$\text{(by induction hypothesis)} \tag{5.26}$$

$$= \frac{1}{i}((D-2(r+i-1))w_{i-1} + (D-2r-(i-1)+1)Rw_{i-2} \quad (Rw_{i-2} = (i-1)w_{i-1}) \tag{5.27}$$

$$= \frac{1}{i}i(D-2r-i+1)w_{i-1} \tag{5.28}$$

$$= (D-2r-i+1)w_{i-1}. \tag{5.29}$$

Claim: $w_0, \ldots, w_d$ is a basis for W.

Pf. Let $W' = \operatorname{Span}\{w_0, \ldots, w_d\}$. Then W' is R and L invariant. So it is $A = R + L$ invariant.

Also it is E^*_i-invariant for every i.

Hence W' is a T-module.

Since W is irreducible, $W' = W$.

As w_i's are orthogonal, they are linearly independent. Note that $w_i \neq 0$ by the definition of d and Lemma 4.1 (iv).

Claim: $d = D - 2r$.

Pf. By (a),

$$0 = (LR - RL - A^*)w_d \tag{5.30}$$

$$= 0 - (D-2r-d+1)Rw_{d-1} - (D-2(r+d))w_d \tag{5.31}$$

$$= -d(D-2r-d+1)w_d - (D-2(r+d))w_d \tag{5.32}$$

$$= (-dD + 2rd + d^2 - d - D + 2r + 2d)w_d \tag{5.33}$$

$$= (d^2 + (2r-D+1)d + 2r - D)w_d \tag{5.34}$$

$$= (d+2r-D)(d+1)w_d. \tag{5.35}$$

Hence $d = D - 2r$.

Therefore, with respect to a bais $w_0, w_1, \ldots, w_d$, $A = L + R$, $w_{-1} = w_{d+1} = 0$,

$$Lw_i = (d-i+1)w_{i-1}, \quad Rw_i = (i+1)w_{i+1}.$$

$$L = \begin{pmatrix} 0 & d & 0 & \cdots & 0 & 0 \\ 0 & 0 & d-1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \cdots & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \qquad R = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 0 & \cdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & 1 \\ 0 & 0 & 0 & \cdots & d & 0 \end{pmatrix}.$$

This completes the proof of (i) and (ii). $\square$

Chapter 6

T-Modules of $H(D,2)$, II

Monday, February 1, 1993

Proof of Theorem 5.1 Continued.

(*iii*) Let $r = r'$,

$w_0, \ldots, w_d$: a basis for W with $w_i \in E_i^* W$, and

$w'_0, \ldots, w'_d$: a basis for W' with $w'_i \in E_i^* W'$.

Then $d = D - 2r = D - 2r' = d'$, and

$$\sigma : W \to W' \quad (w_i \mapsto w'_i)$$

is an isomorsphism of T-modules by (*i*).

If $r \neq r'$, then

$$d = D - 2r \neq D - 2r' = d',$$

hence, $\dim W \neq \dim W'$.

(*iv*) Let W_i be an irreducible T-module with endpoint i. Then

$$\dim E_r^* V = \binom{D}{r} = \sum_{i=0}^{r} \operatorname{mult}(W_i).$$

Hence, we have that

$$\operatorname{mult}(W_r) = \binom{D}{r} - \binom{D}{r-1}$$

by induction on r.

□

Theorem 6.1. *Let $\Gamma = H(D,2)$ with $D \geq 1$. Fix a vertex $x \in X$ and write*

$$E_i^* \equiv E_i^*(x), \quad T = T(x), \text{ and } A^* \equiv \sum_{i=0}^{D}(D-2i)E_i^*.$$

Let W be an irreducible T-module with endpoint r with $0 \leq r \leq D/2$. Then,

(i) W has a basis

$$w_0^*, w_1^*, \dots, w_d^* \quad (d = D-2r), \quad \text{such that } w_i^* \in E_{i+r}W \ (0 \leq i \leq d)$$

with respect to which the matrix corresponding to A^* is

$$\begin{pmatrix} 0 & d & 0 & & & & \\ 1 & 0 & d-1 & & & & \\ 0 & 2 & 0 & & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & 0 & 2 & 0 \\ & & & & d-1 & 0 & 1 \\ & & & & 0 & d & 0 \end{pmatrix}.$$

In particular,

(ii) $E_iA^*E_j = 0$ if $|i-j| \neq 1$ for $0 \leq i,j \leq D$.

Proof. We use the notation,

$$[\alpha, \beta] = \alpha\beta - \beta\alpha \ (= -[\beta, \alpha]).$$

Recall that

(a) $[L, R] = A^*$,

(b) $[A^*, L] = wL$,

(c) $[A^*, R] = -2R$,

and $A = L + R$.

Write $(a) - (c)$ in terms of A and A^*, we have,

$$[A, A^*] = [L, A^*] + [R, A^*] = 2(R - L).$$

$$\begin{cases} R + L &= A \\ R - L &= [A, A^*]/2. \end{cases}.$$

Hence,

$$R = \frac{1}{4}(2A + [A, A^*]) \quad \text{and} \tag{6.1}$$

$$L = \frac{1}{4}(2A - [A, A^*]). \tag{6.2}$$

Now (a), (b) become

$$A^2A^* - 2AA^*A + A^*A^2 - 4A^* = 0 \tag{6.3}$$

$$A^{*2}A - 2A^*AA^* + AA^{*2} - 4A = 0 \tag{6.4}$$

Pf. By (b),

$$\begin{aligned}
2A - AA^* + A^*A &= 4L && (6.5)\\
&= 2[A^*, L] && (6.6)\\
&= A^*\frac{2A - [A, A^*]}{2} - \frac{2A - [A, A^*]}{2}A^* && (6.7)\\
&= A^*A - AA^* + \frac{1}{2}(-A^*AA^* + A^{*2}A + AA^{*2} - A^*AA^*) && (6.8)
\end{aligned}$$

So we have (6.4)

$$A^{*2}A - 2A^*AA^* + AA^{*2} - 4A = 0.$$

By (a),

$$\begin{aligned}
-16A^* &= [2A + [A, A^*], 2A - [A, A^*]] && (6.9)\\
&= (2A + [A, A^*])(2A - [A, A^*]) - (2A - [A, A^*])(2A + [A, A^*]) && (6.10)\\
&= [4A^2 - 2A[A, A^*] + [A, A^*](2A) - [A, A^*]^2 && (6.11)\\
&\quad - 4A^2 - 2A[A, A^*] + [A, A^*](2A) + [A, A^*]^2 && (6.12)\\
&= -4A^2A^* + 4AA^*A + 4AA^*A - 4A^*A^2. && (6.13)
\end{aligned}$$

So,

$$A^2A^* - 2AA^*A + A^*A^2 - 4A^* = 0.$$

Claim: $E_i^*A^*E_j = 0$ if $|i - j| \neq 1$ for $0 \leq i, j \leq D$.

Pf. We have,

$$\begin{aligned}
0 &= E_i(A^2A^* - 2AA^*A + A^*A^2 - 4A^*)E_j && (6.14)\\
&= E_iA^*E_j(\theta_i^2 - 2\theta_i\theta_j + \theta_j^2 - 4) && (6.15)\\
&\quad (AE_j = \theta_jE_j,\ E_iA = (AE_i)^\top = (\theta_iE_i)^\top = \theta_iE_i) && (6.16)\\
&= E_iA^*E_j(\theta_i - \theta_j - 2)(\theta_i - \theta_j + 2) && (6.17)\\
&= E_iA^*E_j(D - 2i - (D - 2j) - 2)(D - 2i - (D - 2j) + 2) && (6.18)\\
&\quad (\theta_k = D - 2k) && (6.19)\\
&= E_iA^*E_j \cdot 4(i - j + 1)(i - j - 1) && (6.20)
\end{aligned}$$

and $i - j + 1 \neq 0$, $i - j - 1 \neq 0$. Hence, $E_i^*A^*E_j = 0$.

Now define "dual raising matrix",

$$R^* = \sum_{i=0}^{D} E_{i+1} A^* E_i.$$

So,

$$R^* E_i V \subseteq E_{i+1} V, \quad (0 \le i \le D,\ E_{D+1} V = 0).$$

Define "dual lowering matrix"

$$L^* = \sum_{i=0}^{D} E_{i-1} A^* E_i.$$

Then

$$L^* E_i V \subseteq E_{i-1} V \quad (0 \le i \le D,\ E_{-1} V = 0).$$

Observe that

$$A^* = \left(\sum_{i=0}^{D} E_i\right) A^* \left(\sum_{j=0}^{D} E_j\right) = L^* + R^*$$

by Claim 1.

Claim 2. We have

(a) $[L^*, R^*] = A$,

(b) $[A, L^*] = 2L^*$,

(c) $[A, R^*] = -2R^*$.

Pf. (b)

$$AL^* - L^* A = \sum_{i=0}^{D} (AE_{i-1} A^* E_i - E_{i-1} A^* E_i A) \tag{6.21}$$

$$= \sum_{i=0}^{D} E_{i-1} A^* E_i (\theta_{i-1} - \theta_i) \tag{6.22}$$

$$(\theta_k = D - 2k, \quad \theta_{i-1} - \theta_i = 2I - 2(i-1) = 2 \tag{6.23}$$

$$= 2L^*. \tag{6.24}$$

(c) Similar.

HS MEMO

$$AR^* - R^* A = \sum_{i=0}^{D} (AE_{i+1} A^* E_i - E_{i+1} A^* E_i A) \tag{6.25}$$

$$= \sum_{i=0}^{D} E_{i+1} A^* E_i (\theta_{i+1} - \theta_i) \tag{6.26}$$

$$= -2R^*. \tag{6.27}$$

(a) We have, by (b), (c)

$$[A, A^*] = [A, L^*] + [A, R^*] = 2(L^* - R^*). \tag{6.28}$$

Since $A^* = L^* + R^*$,

$$R^* = \frac{2A^* + [A^*, A]}{4}, \quad L^* = \frac{2A^* - [A^* - A]}{4}.$$

Now (a) is seen to be equivalent to (6.4) upon evaluation. This proves Claim 2.

HS MEMO

$$[L^*, R^*] = \frac{1}{16}((2A^* - [A^*, A])(2A^* + [A^*, A]) - (2A^* + [A^*, A])(2A^* - [A, A^*])) \tag{6.29}$$

$$= \frac{1}{16}(4{A^*}^2 + 2A^*[A^*, A] - [A^*, A]2A^* - [A^*, A]^2 - 4{A^*}^2 \tag{6.30}$$

$$+ 2A^*[A^*, A] - [A^*, A]2A^* + [A^*, A]^2) \tag{6.31}$$

$$= \frac{1}{4}({A^*}^2 A - 2A^*AA^* + A{A^*}^2) \tag{6.32}$$

$$= A, \tag{6.33}$$

by (6.4).

Now apply same argument as for (6.3), (6.4) of Theorem 5.1 and observe A^* has $D+1$ distinct eigenvalues. So,

$$A^* = \sum_{i=0}^{D} (D - 2i) E_i^*$$

generates

$$M^* = \mathrm{Span}(E_0^*, \dots, E_D^*).$$

Hence, $E_0, \dots, E_D,\ A^*$ generates T.

Take an irreducible T-module W with endpoint r with $0 \le r \le D/2$. Set $t = \min\{i \mid E_i W\}$.

Pick $0 \ne w_0^* \in E_t W$. Set

$$w_i^* = \frac{1}{i!}{R^*}^i w_0^* \in E_{t+i} W \quad \text{for all } i.$$

Then,

$$R^* w_i^* = (i+1) w_{i+1}^* \quad \text{for all } i.$$

By (a), we get by induction, $L^*w_i^* = (D-2t-i+1)w_{i-1}^*$,

$$\begin{aligned} L^*w_i^* &= \frac{1}{i}L^*R^*w_{i-1}^* && (6.34)\\ &= \frac{1}{i}(A+R^*L^*)w_{i-1}^* && (6.35)\\ &= \frac{1}{i}((D-2(t+i-1))w_{i-1}^* + (i-1)(D-2t-i+2)w_{i-1}^*) && (6.36)\\ &= (D-2t-i+1)w_{i-1}^*. && (6.37)\end{aligned}$$

So $\mathrm{Span}(w_0^*, w_1^*, ...)$ is L^*, R^*, A^*-invariant. Hence, $W = \mathrm{Span}(w_0^*, w_1^*, \dots, w_d^*)$, $w_0^*, w_1^*, \dots, w_d^* \neq 0$, $w_i^* = 0$ for every $i > d$ by dimension.

Thus $d = D - 2t$.

Pf.

$$\begin{aligned} (D-2(t+d))w_d^* &= Aw_d^* && (6.38)\\ &= (L^*R^* - R^*L^*)w_d^* && (6.39)\\ &= -(D-2t-d+1)R^*w_{d-1}^* && (6.40)\\ &= -(D-2t-d+1)dw_d^*. && (6.41)\end{aligned}$$

Hence,

$$0 = d^2 + (2t - D - 1 + 2)d - (D - 2t) = (d - D + 2t)(d + 1)$$

So $d = D - 2t$. □

Definition 6.1. For any graph $\Gamma = (X, E)$, pick a vertex $x \in X$, and set $E_i^* \equiv E_i^*(x)$ and $T \equiv T(x)$.

(i) An irreducible T-module W is thin if $\dim E_i^* W \leq 1$ for every i.

(ii) Γ is thin with respet to x, if every irreducible $T(x)$-module is thin,

(iii) An irreducible T-module W is dual thin if $\dim E_i W \leq 1$ for every i.

(iv) Γ is dual thin with respect to x, if every irreducible $T(x)$-module is dual thin.

Observe: $H(D,2)$ is thin, dual thin with respect to each $x \in X$.

Definition 6.2. With above notation, write $D \equiv D(x)$.

(i) An ordering $E_0, E_1, \dots, E_R$ of primitive idempotents of Γ is restricted if E_0 corresponds to the maximal eigenvalue.

Fix a restricted ordering,

(ii) Γ is Q-polynomial with respect to x, above ordering if there exists $A^* \equiv A^*(x)$ such that

(a) $E_0^*V, \dots, E_D^*V$ are the maximal eigenspaces for A^*.

(b) $E_i A^* E_j = 0$ if $|i-j| > 1$ for $0 \leq i, j \leq R$.

Observe $H(D,2)$ is Q-polynomial with respect to the natural ordering of the idempotents and every vetex.

Program. Study graphs that are thin and Q-polynomial with respect to each vertex.

(In fact, thin with respect to x implies dual thin with respect to x.)

Get a situation like $H(D,2)$, where T is generated by A, A^*. Except $\mathrm{sl}_2(\mathbb{C})$ is replaced by a quantum Lie algebra.

Chapter 7

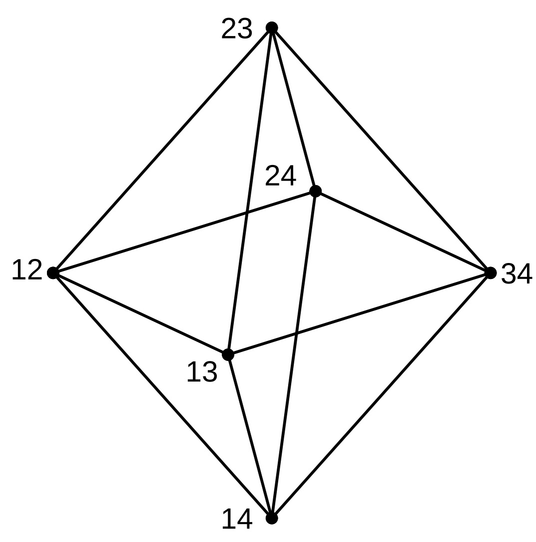
The Johnson Graph $J(D, N)$

Wednesday, February 3, 1993

Definition 7.1. The Johnson graph, $\Gamma = J(D, N)$ $(1 \leq D \leq N-1)$ satisfies

$$X = \{S \mid S \subset \Omega,\ |S| = D\} \quad \text{where} \quad \Omega = \{1, 2, \ldots, N\} \tag{7.1}$$
$$E = \{ST \mid S, T \in X, \quad |S \cap T| = D-1\}. \tag{7.2}$$

Example 7.1. $J(2, 4)$



Note 1. The symmetric group S_N acts on Ω. $S_N \subseteq \mathrm{Aut}(\Gamma)$ acts vertex transitively on Γ.

Note 2. $\Gamma = J(D, N)$ is isomorphic to $\Gamma' = J(N-D, N)$.

$$\Gamma = (X, E) \quad \longrightarrow \quad \Gamma' = (X', E') \tag{7.3}$$
$$X \ni S \quad \longmapsto \quad \bar{S} = \Omega \quad S \in X' \tag{7.4}$$

This correspondence induces an isomorphism of graphs.

Pf.

$$ST \in E \Leftrightarrow |S \cap T| = D - 1 \tag{7.5}$$
$$\Leftrightarrow |\Omega - (S \cup T)| = N - D - 1 \tag{7.6}$$
$$\Leftrightarrow |\bar{S} \cap \bar{T}| = N - D - 1 \tag{7.7}$$
$$\Leftrightarrow \bar{S}\bar{T} \in E' \tag{7.8}$$

Hence, without loss of generality, assume

$$D \leq N/2 \quad \text{for} \quad J(D, N).$$

We will need the eigenvalues of $J(D, N)$ for certain problem later in the course. We can get these eigenvalues from our study of $H(D, 2)$.

Lemma 7.1. *The eigenvalues for $J(D, N)$ with $1 \leq D \leq N/2$ are give by*

$$\theta_i = (N - D - i)(D - i) - i \quad (0 \leq i \leq D), \tag{7.9}$$
$$m_i = \binom{N}{i} - \binom{N}{i-1}. \tag{7.10}$$

Proof. Let

$$\Gamma_J \equiv J(D, N) = (X_J, E_J) \tag{7.11}$$
$$\Gamma_H \equiv H(N, 2) = (X_H, E_H). \tag{7.12}$$

Set $x \equiv 11 \cdots 1 \in X_H$.

Define $\tilde{\Gamma} \equiv (\tilde{X}, \tilde{E})$, where

$$\tilde{X} = \{y \in X_H \mid \partial_H(x, y) = D\} \quad \partial_H : \text{distance in } \Gamma_H \tag{7.13}$$
$$\tilde{E} = \{yz \in X_H \mid \partial_H(y, z) = 2\}. \tag{7.14}$$

Observe

$$X_J \quad \to \quad \tilde{X} \tag{7.15}$$
$$S \quad \mapsto \quad \hat{S}, \tag{7.16}$$

where

$$\hat{S} = a_1 \cdots a_N, \quad a_i = \begin{cases} -1 & \text{if } i \in S \\ 1 & \text{if } i \notin S \end{cases}$$

induces an isomorphism of graphs $\Gamma_J \to \tilde{\Gamma}$.

Pf.

$$\begin{aligned} ST \in E_J &\Leftrightarrow |S \cap T| = D-1 && (7.17)\\ &\Leftrightarrow \partial_H(\hat{S}, \hat{T}) = 2 && (7.18)\\ &\Leftrightarrow (\hat{S}, \hat{T}) \in \tilde{E}. && (7.19)\end{aligned}$$

Identify, Γ_J with $\tilde{\Gamma}$. Then the standard module V_J of Γ_J becomes $\tilde{V} = E_D^* V_H$, where V_H is the standard module of Γ_H, and $E_D^* \equiv E_D^*(x)$.

Let R be the raising matrix with respect to x in Γ_H, and

let L be the lowering matrix with respect to x in Γ_H.

Recall

$$(RL - DE_D^*)|_{\tilde{V}}$$

is the adjacency map in $\tilde{\Gamma}$.

To find eigenvalues of $\tilde{A}$, pick any irreducible $T(x)$-module W with the endpoint $r \le D$. Then by Theorem 5.1

$$\operatorname{diam}(W) = N - 2r.$$

Let $w_0, w_1, \ldots, w_{N-2r}$ denote a basis for W as in Theorem 5.1. Then,

$$w_{D-r} \in E_D^* W \subseteq \tilde{V}.$$

Observe:

$$\begin{aligned} \tilde{A} w_{D-r} &= RLw_{D-r} - DE_D^* w_{D-r} && (7.20)\\ &= R(N-2r-D+r+1)w_{D-r-1} - Dw_{D-r} && (7.21)\\ &= ((N-D-r+1)(D-r)-D)w_{D-r}. && (7.22)\end{aligned}$$

Note that this is valid for $D = r$ as well.

Hence,

$$\tilde{A} w_{D-r} = ((N-D-r)(D-r)-r)w_{D-r}.$$

Let

$$V_H = \sum W \quad \text{(direct sum of irreducible } T(x)\text{-modules).}$$

Then,

$$\begin{aligned} V_J &= E_D^* V_H && (7.23)\\ &= \sum_{W: r(W) \le D} E_D^* W && (7.24)\\ &= \text{a direct sum of 1 dimensional eigenspaces for } \tilde{A}. && (7.25)\end{aligned}$$

The eigenspace for eigenvalue

$$(N-D-r)(D-r)-r \quad \text{(monotonously decreasing with respec to } r\text{)}$$

appears with multiplicity

$$\binom{N}{r} - \binom{N}{r-1}$$

in this sum by Theorem 5.1 (*iv*). $\square$

Theorem 7.1. *Let $\Gamma = (X, E)$ be any graph. For a fixed vertex $x \in X$, let*

$$E_i^* \equiv E_i^*(x), \quad T \equiv T(x), \quad D \equiv D(x), \text{ and } K = \mathbb{C}.$$

Then we have the following implications of conditions:

$$TH \Leftrightarrow C \Leftarrow S \Leftarrow G,$$

where

(TH) Γ is thin with respect to x.

(C) $E_i^* T E_i^*$ is commutative for every i, $(0 \leq i \leq D)$.

(S) $E_i^* T E_i^*$ is symmetric for every i, $(0 \leq i \leq D)$.

(G) For every $y, z \in X$ with $\partial(x, y) = \partial(x, z)$, there exists $g \in \mathrm{Aut}(\Gamma)$ such that

$$gx = x, \ gy = z, \ gz = y.$$

Proof.

(TH) $\Rightarrow$ (C)

Fix i with $0 \leq i \leq D$. Let

$V = \sum W$. The standard module written as a direct sum of irreducible T-modules.

Then,

$$E_i^* V = \sum E_i^* W. \text{ The direct sum of 1-dimensional } E_i^* T E_i^*\text{-modules.}$$

Since $\dim E_i^* W = 1$, for $a, b \in E_i^* T E_i^*$, $ab - ba_{|E_i^* W} = 0$. Hence $ab - ba = 0$.

(C) $\Rightarrow$ (TH)

Suppose $\dim E_i^* W \geq 2$ for some irreducible T-module W with some i with $1 \leq i \leq D$.

Claim 1. $E_i^* W$ is an irreducible $E_i^* T E_i^*$-module.

Proof of Claim 1. Suppose

$$0 \subsetneq U \subsetneq E_i^* W,$$

where U is an $E_i^*TE_i^*$-module. Then by the irreducibility,

$$TU = W.$$

So,

$$U \supseteq E_i^*TE_i^*U = E_i^*TU = E_i^*W.$$

This is a contradiction.

Claim 2. Each irreducible $S = E_i^*TE_i^*$-module U has dimension 1. In particular, Γ is thin with respect to x.

Proof of Claim 2. Pick

$$0 \neq a \in E_i^*TE_i^*.$$

Since $\mathbb{C}$ is algebraically closed, a has an eigenvector $w \in U$ with eigenvalue θ. Then,

$$\begin{aligned}
(a - \theta I)U &= (a - \theta I)Sw && (7.26)\\
&= S(a - \theta I)w && (7.27)\\
&= 0. && (7.28)
\end{aligned}$$

Hence,

$$a_{|U} = \theta I_{|U} \quad \text{for all} \quad a \in S.$$

Thus each 1 dimensional subspace of U is an S-module. We have

$$\dim U = 1.$$

By Claim 1 and Claim 2, we have (TH).

$\square$

HS MEMO

Claim 1 shows the following: *If W is an irreducible T-module, then E_i^*W is either 0 or an irreducible $E_i^*TE_i^*$-module.*

Chapter 8

Thin Graphs

Friday, February 5, 1993

Proof of Theorem 7.1 continued.

(S) $\Rightarrow$ (C)

Fix i and pick $a, b \in E_i^* T E_i^*$.

Since a, b and ab are symmetric,

$$ab = (ab)^\top = b^\top a^\top = ba.$$

Hence $E_i^* T E_i^*$ is commutative.

(G) $\Rightarrow$ (S)

Fix i and pick $a \in E_i^* T E_i^*$. Pick vertices $y, z \in X$.

We want to show that

$$a_{yz} = a_{zy}.$$

We may assume that

$$\partial(x, y) = \partial(x, z) = i,$$

otherwise

$$a_{yz} = a_{zy} = 0.$$

By our assumption, there exists $g \in G$ such that

$$g(y) = z, \quad g(z) = y, \quad g(x) = x.$$

Let $\hat{g}$ denote the permutation matrix representing g, i.e.,

$$\hat{g}\hat{y} = \widehat{g(y)} \quad \text{for all} \quad y \in X, \quad \hat{y} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow y.$$

If $g \in \mathrm{Aut}(\Gamma)$, then

$$\hat{g}A = A\hat{g} \quad \text{(Exercise)}.$$

Also, we have

$$\hat{g}E_j^* = E_j^*\hat{g} \quad (0 \le j \le D),$$

since

$$\partial(x, y) = \partial(g(x), g(y)) = \partial(x, g(y)).$$

Hence, $\hat{g}$ commutes with each element of T. We have

$$a_{yz} = (\hat{g}^{-1} a \hat{g})_{yz}, \quad (\hat{g})_{yz} = \begin{cases} 1 & g(z) = y \\ 0 & \text{else.} \end{cases} \tag{8.1}$$

$$= \sum_{y',z'} (\hat{g}^{-1})_{yy'} a_{y'z'} \hat{g}_{z'z} \tag{8.2}$$

$$\text{(zero except for } g^{-1}(y') = y,\ g(z) = z'.) \tag{8.3}$$

$$= a_{g(y)g(z)} \tag{8.4}$$

$$= a_{zy}. \tag{8.5}$$

This proves Theorem 7.1.

□

Open Problem: Find all the graphs that satisfy the condition (G) for every vertex x.

$H(N, 2)$ is one example, because

$$\mathrm{Aut}\Gamma_{1\cdots 1} \simeq S_\Omega, \quad x = (1 \cdots 1), \quad \Gamma_i(x) = \{\hat{S} \mid |S| = i\}.$$

Property (G) is clearly related to the distance-transitive property.

Definition 8.1. Let $\Gamma = (X, E)$ be any graph. Γ with $G \subseteq \mathrm{Aut}(\Gamma)$ is said to be distance-transitive (or two-point homogeneous), whenever

$$\text{for all } x, x', y, y' \in X \ \text{ with } \partial(x, y) = \partial(x', y'),$$

there exists $g \in G$ such that

$$g(x) = x', \quad g(y) = y'.$$

(This means G is as close to being doubly transitive as possible.)

Lemma 8.1. *Suppose a graph $\Gamma = (X, E)$ satisfies the property (G) = (G(x)) for every $x \in X$. Then,*

(i) *either*
(ia) *Γ is vertex transitive; or*
(iia) *Γ is bipartite ($X = X^+ \cup X^-$) with X^+, X^- each an orbit of* $\mathrm{Aut}(\Gamma)$.

(ii) *if (ia) holds, then Γ is distance-transitive.*

Proof. (i) Claim. Suppose $y, z \in X$ are conneced by a path of even length. Then y, z are in the same orbit of $\mathrm{Aut}(\Gamma)$.

Pf of Claim. It suffices to assume that the path has lenght 2, $y \sim w \sim z$.

Now $\partial(y, w) = \partial(w, z) = 1$. So there exits $g \in \mathrm{Aut}(\Gamma)$ such that

$$gw = w, \quad gy = z, \quad gz = y.$$

This proves Claim.

Fix $x \in X$. Now suppose that Γ is not vertex transitive, and we shall show (ib).

Observe that $X = X^+ \cup X^-$, where

$$X^+ = \{y \in X \mid \text{there exists a path of even length connecting } x \text{ and } y\}, \quad (8.6)$$
$$X^- = \{y \in X \mid \text{there exists a path of odd length connecting } x \text{ and } y\}. \quad (8.7)$$

Also, X^+ is contained in an orbit O^+ of $\mathrm{Aut}(\Gamma)$, and X^- is contained in an orbit O^- of $\mathrm{Aut}(\Gamma)$.

Now $O^+ \cap O^- = \emptyset$ (else $O^+ = O^- = X$ and vertex transitive). So, $X = O^+$, and $X^- = O^-$.

Also $X^+ \cup X^- = X$ is a bipartition by construction.

(ii) Fix x, y, x', y' with $\partial(x, y) = \partial(x', y')$.

By vertex transitivity, there exists an element

$$g_1 \in G \text{ such that } g_1 x = x'.$$

Observe that

$$\partial(x', y') = \partial(x, y) = \partial(g_1 x, g_1 y) = \partial(x', g_1 y).$$

Hence, there exisits an element

$$g_2 \in G \text{ such that } g_1 x' = x', g_2 y' = g_1 y, g_2 g_1 y = y'$$

by $(\mathrm{G}(x'))$ property.

Set $g = g_2 g_1$. Then

$$gx = x', gy = y'$$

by construction. □

The following graphs $\Gamma = (X, E)$ are vertex transitive, and satisfy the property $(\mathrm{G}(x))$ for all $x \in X$.

$$J(D, N), \quad H(D, r), \quad J_q(D, N),$$

where

$H(D,r)$:

$$X = \{a_1 \cdots a_D \mid a_i \in F, 1 \le i \le D\} \tag{8.8}$$
$$F: \text{ any set of cardinality } r \tag{8.9}$$
$$E = \{xy \mid y, x \in X,\ x \text{ and } y \text{ differ in exactly one coordiate}\}. \tag{8.10}$$

$J_q(D,N)$:

$$X = \text{the set of all } D\text{-dimensional subspaces of } N\text{-dimensional vector space over } GF(q). \tag{8.11}$$
$$F: \text{ any set of cardinality } r \tag{8.12}$$
$$E = \{xy \mid y, x \in X,\ \dim(x \cap y) = D - 1\}. \tag{8.13}$$

The following graph is distance-transitive but does not satisify (G(x)) for any $x \in G$.

$H_q(D,N)$:

$$X = \text{the set of all } D \times N \text{ matrices with entries in } GF(q). \tag{8.14}$$
$$E = \{xy \mid y, x \in X,\ \mathrm{rank}(x - y) = 1\}. \tag{8.15}$$

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$H(D,r)$: $G = S_r \mathrm{wr} S_D$, $G_x = S_{r-1} \mathrm{wr} S_D$,

For $x, y \in X$ with $\partial(x,y) = \partial(x,z) = i$,

$$Y = \{j \in \Omega \mid x_j \ne y_j\} \leftrightarrow Z = \{j \in \Omega \mid x_j \ne z_j\} \tag{8.16}$$
$$(y_{j_1}, \dots, y_{j_i}) \leftrightarrow (z_{\ell_1}, \dots, z_{\ell_i}) \tag{8.17}$$

$J(D,N)$: $G = S_N$, $G_x = S_D \times S_{N-D}$.

$$X \cap Y \leftrightarrow X \cap Z \tag{8.18}$$
$$(\Omega\ \ X) \cap Y \leftrightarrow (\Omega\ \ X) \cap Z. \tag{8.19}$$

$J_q(D,N)$:

$$X \cap Y \leftrightarrow X \cap Z.$$

The theory of a single thin irreducible T-module.

Let $\Gamma = (X, E)$ be any graph.

$$M = \text{Bose-Mesner algebra over } K/\mathbb{C} \text{ generated by the adjacency matrix } A. \tag{8.20}$$

$$= \text{Span}(E_0, \dots, E_R). \tag{8.21}$$

M acts on the standard module $V = \mathbb{C}^{|X|}$.

Fix $x \in X$, let $D \equiv D(x)$ be the x-diameter, and $k = k(x)$ be the valency of x.

Chapter 9

Thin T-Module, I

Monday, February 8, 1993

Let $\Gamma = (X, E)$ be any graph.

M: Bose-Mesner algebra over $K/\mathbb{C}$ generated by the adjacency matrix A.

$M = \operatorname{Span}(E_0, \dots, E_R)$.

M acts on the standard module $V = \mathbb{C}^{|X|}$.

Fix $x \in X$, let $D \equiv D(x)$ be the x-diameter, and $k = k(x)$ be the valency of x.

Definition 9.1. Pick $x \in X$ and write $E_i^* \equiv E_i^*(x)$ and $T \equiv T(x)$.

Let W be an irreducible thin T-module with endpoint r, diameter d.

Let $a_i = a_i(W) \in \mathbb{C}$ satisfying

$$E_{r+i}^* A E_{r+i}^*|_{E_{r+i}^* W} = a_i 1|_{E_{r+i}^*} \quad (0 \le i \le d).$$

Let $x_i = x_i(W) \in \mathbb{C}$ satisfying

$$E_{r+i-1}^* A E_{r+i}^* A E_{r+i-1}^*|_{E_{r+i-1}^* W} = x_i 1|_{E_{r+i-1}^*} \quad (0 \le i \le d).$$

Lemma 9.1. *With above notation, the following hold.*

(i) $a_i \in \mathbb{R} \quad (0 \le i \le d)$.

(ii) $x_i \in \mathbb{R}^{>0} \quad (0 \le i \le d)$.

(iii) Pick $0 \ne w_0 \in E_r^* W$. *Set* $w_i = E_{r+i}^* A^i w_0$ *for all* i. *Then*

(iiia) $w_0, w_1, \dots, w_d$ *is a basis for* W, $w_{-1} = w_{d+1} = 0$.

(iiib) $A w_i = w_{i+1} + a_i w_i + x_i w_{i-1} \quad (0 \le i \le d)$.

(iv) Define $p_0, p_1, \dots, p_{d+1} \in \mathbb{R}[\lambda]$ *by*

$$p_0 = 1, \quad \lambda p_i = p_{i+1} + a_i p_i + x_i p_{i-1} \quad (0 \le i \le d), \quad p_{-1} = 0.$$

(iva) $p_i(A)w_0 = w_i, \quad (0 \le i \le d+1)$.
(ivb) p_{d+1} *is the minimal polynomial of* $A|_W$.

Proof. *(i)* a_i is an eigenvalue of a real symmetric matrix $E^*_{r+i}AE^*_{r+i}$.

(ii) x_i is an eigenvalue of a real symmetrix matrix $B^\top B$, where

$$B = E^*_{r+i}AE^*_{r+i-1}.$$

Hence, $x_i \in \mathbb{R}$.

Since $B^\top B$ is positive semidefinite,

$$x_i \ge 0.$$

Pf. If $B^\top Bv = \sigma v$ for some $\sigma \in \mathbb{R}$, $v \in \mathbb{R}^m \quad \{0\}$, then

$$0 \le \|Bv\|^2 = v^\top B^\top Bv = \sigma v^\top v = \sigma\|v\|^2, \quad \|v\|^2 > 0.$$

Hence, $\sigma \ge 0$.

Moreover, $x_i \ne 0$ by Lemma 4.1 *(iv)*.

(iiia) Observe

$$w_i = E^*_{r+i}AE^*_{r+i-1}w_{i-1} \quad (1 \le i \le d).$$

So $w_i \ne 0 \quad (0 \le i \le d)$ by Lemma 4.1 *(iv)*.

Hence,

$$W = \mathrm{Span}(w_0, \dots, w_d)$$

by Lemma 4.1. *(iii)*.

(iiib) We have that

$$\begin{aligned} Aw_i &= E^*_{r+i+1}Aw_i + E^*_{r+i}Aw_i + E^*_{r+i-1}Aw_i && (9.1)\\ &= w_{i+1} + E^*_{r+i}AE^*_{r+i}w_i + E^*_{r+i-1}AE^*_{r+i}AE^*_{r+i-1}w_{i-1} && (9.2)\\ &= w_{i+1} + a_i w_i + x_i w_{i-1}. && (9.3)\end{aligned}$$

(iva) Clear for $i = 0$. Assume it is valid for $0, \dots, i$.

$$p_{i+1}(A)w_0 = (A - a_i I)w_i - x_i w_{i-1} = w_{i+1}.$$

(ivb) By definition,

$$p_{d+1}(A)w_0 = 0.$$

Moreover, $p_{d+1}(A)W = 0$ because of the following.

For every $w \in W$, write

$$w = \sum_{i=0}^{d} \alpha_i w_i \tag{9.4}$$

$$= \sum_{i=0}^{d} \alpha_i p_i(A) w_0 \qquad \text{for some } \alpha_i \in \mathbb{C} \tag{9.5}$$

$$= p(A) w_0 \qquad \text{for some } p \in \mathbb{C}[\lambda]. \tag{9.6}$$

Hence,

$$p_{d+1}(A)w = p_{d+1}(A)p(A)w_0 \tag{9.7}$$

$$= p(A)p_{d+1}(A)w_0 \tag{9.8}$$

$$= 0. \tag{9.9}$$

Note that p_{d+1} is the minimal polynomial.

Pf. Suppose $q(A)W = 0$ for some $0 \neq q \in \mathbb{C}[\lambda]$ with $\deg q < \deg p_{d+1} = d+1$. Then,

$$q = \sum_{i=0}^{d} \beta_i p_i \quad \text{for some } \beta_i \in \mathbb{C}.$$

We have,

$$0 = q(A)w_0 = \sum_{i=0}^{d} \beta_i w_i.$$

Hence $\beta_0 = \cdots = \beta_d = 0$ by $(iiia)$. Thus $q = 0$, and a contradiction. □

Corollary 9.1. *Let Γ, W, r, d be as above. Then*

(i) W is dual thin, that is,

$$\dim E_i W \leq 1 \quad (1 \leq i \leq d).$$

(ii) $d = |\{i \mid E_i W \neq 0\}| - 1$.

Proof. (i) Set as in Lemma 9.1,

$$w_i = p_i(A) w_0 \in E^*_{r+i} W.$$

Then $w_0, w_1, \dots, w_d$ is a basis for W. We have

$$W = M w_0.$$

So,

$$E_i W = E_i M w_0 = \mathrm{Span}(E_i w_0).$$

Thus,

$$\dim E_i W = \begin{cases} 1 & \text{if } E_i w_0 \neq 0, \\ 0 & \text{if } E_i w_0 = 0. \end{cases}$$

In particular,

$$\dim E_i^* W \leq 1.$$

(*ii*) Immediate as

$$\dim W = d + 1.$$

This proves the lemma. □

Lemma 9.2. *Given an irreducible $T(x)$-module W with endpoint $r = r(W)$, diameter $d = d(W)$. Write*

$$x_i = x_i(W)\ (0 \leq i \leq d), \quad w_i = p_i(A) w_0 \in E_{r+i}^* W\ (0 \leq i \leq d), \quad 0 \neq w_0 \in E_r^* W.$$

Then,

$$\frac{\|w_i\|^2}{\|w_0\|^2} = x_1 x_2 \cdots x_i \quad (1 \leq i \leq d).$$

Proof. It suffices to show that

$$\|w_i\|^2 = x_i \|w_{i-1}\|^2 \quad (1 \leq i \leq d).$$

Recall by Lemma 9.1 (*iiib*) that

$$A w_j = w_{j+1} + a_j w_j + x_j w_{j-1} \quad (0 \leq j \leq d), \quad w_{-1} = w_{d+1} = 0.$$

Now observe,

$$\begin{aligned} \langle w_{i-1}, A w_i \rangle &= \langle w_{i-1}, w_{i+1} + a_i w_i + x_i w_{i-1} \rangle && (9.10) \\ &= \overline{x_i} \|w_{i-1}\|^2 && (9.11) \\ &= x_i \|w_{i-1}\|^2. && (9.12) \end{aligned}$$

by Lemma 9.1 (*ii*). Also,

$$\begin{aligned} \langle w_{i-1}, A w_i \rangle &= \langle A w_{i-1}, w_i \rangle \quad (\text{since } \bar{A}^\top = A) && (9.13) \\ &= \langle w_i + a_{i-1} w_{i-1} + x_{i-1} w_{i-2}, w_i \rangle && (9.14) \\ &= \|w_i\|^2. && (9.15) \end{aligned}$$

This proves the lemma. □

Definition 9.2. Let W be an irreducible thin $T(x)$ module with endpoint r, $E_i^* \equiv E_i^*(x)$.

The measure $m = m_W$ is the function

$$m : \mathbb{R} \to \mathbb{R}$$

such that

$$m(\theta) = \begin{cases} \frac{\|E_i w\|^2}{\|w\|^2} & \text{where } 0 \neq w \in E_r^* W \\ & \text{ if } \theta = \theta_i \text{ is an eigenvalue for } \Gamma, \\ 0 & \text{if } \theta \text{ is not an eigenvalue for } \Gamma. \end{cases}$$

Chapter 10

Thin T-Module, II

Wednesday, February 10, 1993

Let $\Gamma = (X, E)$ be any graph.

Fix a vertex $x \in X$. Let $E_i^* \equiv E_i^*(x)$, $T \equiv T(x)$, the subconstituent algebra over $\mathbb{C}$, and $V = \mathbb{C}^{|X|}$ the standard module.

Lemma 10.1. *With above notation, let W denote a thin irreducible $T(x)$-module with endpoint r and diameter d. Let*

$$a_i = a_i(W) \quad (0 \le i \le d) \tag{10.1}$$
$$x_i = x_i(W) \quad (1 \le i \le d) \tag{10.2}$$
$$p_i = p_i(W) \quad (0 \le i \le d+1) \tag{10.3}$$

be from Lemma 9.1, and measure $m = m_W$. Then,

(i) $p_0, \dots, p_{d+1}$ *are orthogonal with respect to m, i.e.,*

$$\sum_{\theta \in \mathbb{R}} p_i(\theta) p_j(\theta) m(\theta) = \delta_{ij} x_1 x_2 \cdots x_i \quad (0 \le i, j \le d+1) \text{ with } x_{d+1} = 0.$$

(ia) $\displaystyle\sum_{\theta \in \mathbb{R}} p_i(\theta)^2 m(\theta) = x_1 \cdots x_i \quad (0 \le i \le d).$

(iia) $\displaystyle\sum_{\theta \in \mathbb{R}} m(\theta) = 1.$

(iiia) $\displaystyle\sum_{\theta \in \mathbb{R}} p_i(\theta)^2 \theta m(\theta) = x_1 \cdots x_i a_i \quad (0 \le i \le d).$

Proof. Pick $0 \ne w_0 \in E_r^* W$. Set

$$w_i = p_i(A) w_0 \in E_{r+i}^* W.$$

Since E_i^*W and E_j^*W are orthogonal if $i \neq j$,

$$\delta_{ij}\|w_i\|^2 = \langle w_i, w_j \rangle \tag{10.4}$$
$$= \langle p_i(A)w_0, p_j(A)w_0 \rangle \tag{10.5}$$
$$= \left\langle p_i(A)\left(\sum_{\ell=0}^{R} E_\ell\right) w_0, p_j(A)\left(\sum_{\ell=0}^{R} E_\ell\right) w_0 \right\rangle \tag{10.6}$$
$$= \left\langle \sum_{\ell=0}^{R} p_i(\theta_\ell)E_\ell w_0, \sum_{\ell=0}^{R} p_j(\theta_\ell)E_\ell w_0 \right\rangle \qquad (\text{as } AE_j = \theta_j E_j) \tag{10.7}$$
$$= \sum_{\ell=0}^{R} p_i(\theta_\ell)\overline{p_j(\theta_\ell)}\|E_\ell w_0\|^2 \tag{10.8}$$
$$(\text{as } \ p_j \in \mathbb{R}[\lambda], \quad \theta_\ell \in \mathbb{R}, \quad m(\theta_i)\|w_0\|^2 = \|E_i w_0\|^2) \tag{10.9}$$
$$= \sum_{\theta \in \mathbb{R}} p_i(\theta)p_j(\theta)m(\theta)\|w_0\|^2. \tag{10.10}$$

Now we are done by Lemma 9.2 as

$$\|w_i\|^2 = \|w_0\|^2 x_1 x_2 \ldots x_i.$$

For (ia), set $i = j$, and for (ib), set $i = j = 0$.

(ii) We have

$$\langle w_i, Aw_i \rangle = \langle w_i, w_{i+1} + a_i w_i + x_i w_{i-1} \rangle \tag{10.11}$$
$$= \overline{a_i}\|w_i\|^2 \tag{10.12}$$
$$= a_i x_1 \cdots x_i \|w_0\|^2, \tag{10.13}$$

as $a_i \in \mathbb{R}$ by Lemma 9.1.

Also,

$$\langle w_i, Aw_i \rangle = \langle p_i(A)w_0, Ap_i(A)w_0 \rangle \tag{10.14}$$
$$= \left\langle p_i(A)\left(\sum_{\ell=0}^{R} E_\ell\right) w_0, Ap_i(A)\left(\sum_{\ell=0}^{R} E_\ell\right) w_0 \right\rangle \qquad (\text{ as in } (i)) \tag{10.15}$$
$$= \sum_{\ell=0}^{D} p_i(\theta_\ell)^2 \theta_\ell \|E_\ell w_0\|^2 \tag{10.16}$$
$$= \sum_{\theta \in \mathbb{R}} p_i(\theta)^2 \theta m(\theta)\|w_0\|^2. \tag{10.17}$$

Thus, we have (ii). $\square$

Lemma 10.2. *With above notation, let W be a thin irreducible $T(x)$-module with measure m. Then m determines diameter $d(W)$,*

$$a_i = a_i(W) \quad (0 \le i \le d) \tag{10.18}$$
$$x_i = x_i(W) \quad (1 \le i \le d) \tag{10.19}$$
$$p_i = p_i(W) \quad (0 \le i \le d+1). \tag{10.20}$$

Proof. Note that $d+1$ is the number of $\theta \in \mathbb{R}$ such that $m(\theta) \ne 0$. Hence m determines d.

Apply (ia), (ii) of Lemma 10.1.

$$\sum_{\theta\in\mathbb{R}} m(\theta) = 1 \qquad p_0 = 1. \tag{10.21}$$
$$\sum_{\theta\in\mathbb{R}} \theta m(\theta) = a_0 \qquad p_1 = \lambda - a_0 \tag{10.22}$$
$$\sum_{\theta\in\mathbb{R}} p_1(\theta)^2 m(\theta) = x_1 \tag{10.23}$$
$$\sum_{\theta\in\mathbb{R}} p_1(\theta)^2 \theta m(\theta) = x_1 a \qquad \to a_1 \tag{10.24}$$
$$p_2 = (\lambda - a_1)p_1 - x_1 p_0 \tag{10.25}$$
$$\sum_{\theta\in\mathbb{R}} p_2(\theta)^2 m(\theta) = x_1 x_2 \qquad \to x_2 \tag{10.26}$$
$$\sum_{\theta\in\mathbb{R}} p_2(\theta)^2 \theta m(\theta) = x_1 x_2 a_2 \qquad \to a_2 \tag{10.27}$$
$$p_3 = (\lambda - a_2)p_2 - x_2 p_1 \tag{10.28}$$
$$\vdots \tag{10.29}$$
$$\sum_{\theta\in\mathbb{R}} p_d(\theta)^2 m(\theta) = x_1 x_2 \cdots x_d \qquad \to x_d \tag{10.30}$$
$$\sum_{\theta\in\mathbb{R}} p_d(\theta)^2 \theta m(\theta) = x_1 x_2 \cdots x_d a_d \qquad \to a_d \tag{10.31}$$
$$p_{d+1} = (\lambda - a_d)p_d - x_d p_{d-1}. \tag{10.32}$$
$$\tag{10.33}$$

This proves the assertions. □

Corollary 10.1. *With above notation, let W, W' denote thin irreducible $T(x)$-modules. The following are equivalent.*

(i) W, W' are isomorphic as T-modules.

(ii) $r(W) = r(W')$ and $m_W = m_{W'}$.

(iii) $r(W) = r(W')$, $d(W) = d(W')$, $a_i(W) = a_i(W')$ and $x_i(W) = x_i(W')$ $(0 \le i \le d)$.

Proof. $(i) \Rightarrow (iii)$ Write $r \equiv r(W)$, $r' \equiv r(W')$, $d \equiv d(W)$, $d' \equiv d(W')$, $a_i \equiv a_i(W)$, $a'_i \equiv a_i(W')$, $x_i \equiv x_i(W)$ and $x'_i \equiv x_i(W')$.

Let $\sigma : W \to W'$ denote an isomorphism of T-modules. (See Definition 5.1.)

For every i,

$$\sigma E^*_i W = E^*_i \sigma W = E^*_i W'.$$

So, $r = r'$ and $d = d'$.

To show $a_i = a'_i$, pick $w \in E^*_{r+i}W \quad \{0\}$. Then,

$$E^*_{r+i}AE^*_{r+i}\sigma(W) = \sigma(E^*_{r+i}AE^*_{r+i}w) = \sigma(a_i w) = a_i \sigma(w),$$

and $\sigma w \neq 0$. So,

$$a_i = \text{eigenvalue of } E^*_{r+i}AE^*_{r+i} \text{ on } E^*_{r+i}W \tag{10.34}$$

$$= a'_i. \tag{10.35}$$

It is similar to show $x = x'$.

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Pick $w \in E^*_{r+i-1}W \quad \{0\}$, then

$$E^*_{r+i-1}AE^*_{r+i}AE^*_{r+i-1}\sigma(W) = \sigma(E^*_{r+i-1}AE^*_{r+i}AE^*_{r+i-1}w) = x_i\sigma(w).$$

Hence, x_i is the eigenvalue of $E^*_{r+i-1}AE^*_{r+i}AE^*_{r+i-1}$ on $E^*_{r+i-1}W = x'_i$.

$(iii) \Rightarrow (i)$

Pick $0 \neq w_0 \in E^*_r W$, $0 \neq w'_0 \in E^*_r W'$. Let p_i be in Lemma 9.1, and set

$$w_i = p_i(A)w_0 \in E^*_{r+i}W \quad (0 \leq i \leq d), \tag{10.36}$$

$$w'_i = p'_i(A)w'_0 \in E^*_{r+i}W \quad (0 \leq i \leq d). \tag{10.37}$$

Define a linear transformation,

$$\sigma : W \to W' \quad (w_i \mapsto w'_i).$$

Since $\{w_i\}$ and $\{w'_i\}$ are bases with $d = d'$, σ is an isomorphism of vector spaces.

We need to show

$$a\sigma = \sigma a \quad (\text{for all } \ a \in T).$$

Take $a = E^*_j$ for some j $(0 \leq j \leq d(x))$. Then for all i, we have

$$E^*_j \sigma w_i = E^*_j w'_i = \delta_{ij} w'_i,$$

$$\sigma E^*_j w_i = \delta_{ij}\sigma(w_i) = \delta_{ij} w'_i.$$

$$E^*_j \sigma w_i = \sigma E^*_j w_i?$$

Take an adjacency matrix A of a. Then,

$$A\sigma w_i = Aw'_i = w'_{i+1} + a'_i w'_i + x'_i w'_{i-1} = \sigma(w_{i+1} + a_i w_i + x_i w_{i-1}) = \sigma A w_i.$$

$(ii) \Rightarrow (iii)$ Lemma 10.2.

$(iii) \Rightarrow (ii)$ Given d, a_i, x_i, we can compute the polynomial sequence

$$p_0, p_1, \dots, p_{d+1}$$

for W.

Show $p_0, p_1, \dots, p_{d+1}$ determines $m = m_W$. Set

$$\Delta = \{\theta \in \mathbb{R} \mid p_{d+1}(\theta) = 0\}.$$

Observe: $|\Delta| = d+1$. See 'An Introcuction to Interlacing'.

$m(\theta) = 0$ if $\theta \notin \Delta \quad (\theta \in \mathbb{R})$. So it suffices to find $m(\theta)$, $\theta \in \Delta$.

By Lemma 10.1 (i),

$$\begin{cases} \sum_{\theta\in\Delta} m(\theta)p_0(\theta) & = 1, \\ \sum_{\theta\in\Delta} m(\theta)p_1(\theta) & = 0, \\ \qquad\vdots & \\ \sum_{\theta\in\Delta} m(\theta)p_d(\theta) & = 0. \end{cases}$$

$d+1$ linear equation with $d+1$ unknowns $m(\theta)$ $(\theta \in \Delta)$.

But the coefficient matrix is essentially Vander Monde (since $\deg p_i = i$). Hence the system is nonsingular and there are unique values for $m(\theta)$ $(\theta \in \Delta)$. $\square$

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$$\begin{pmatrix} \theta - a_0 & -1 & \cdots & 0 & 0 \\ -x_1 & \theta - a_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \theta - a_{d-1} & -1 \\ 0 & 0 & \cdots & -x_d & \theta - a_d \end{pmatrix} \begin{pmatrix} p_0(\theta) \\ \vdots \\ \vdots \\ \vdots \\ p_d(\theta) \end{pmatrix} = 0,$$

where θ is an eigenvalue of a diagonalizable matrix

$$L = \begin{pmatrix} a_0 & 1 & \cdots & 0 & 0 \\ x_1 & a_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{d-1} & 1 \\ 0 & 0 & \cdots & x_d & \theta a_d \end{pmatrix}$$

with multiplicity $\dim(\mathrm{Ker}(\theta I - L) = 1)$.

Chapter 11

Examples of T-Module

Friday, February 12, 1993

Let $\Gamma = (X, E)$ be a connected graph.

Let θ_0 be the maximal eigenvalue of Γ, and δ its corresponding eigenvector.

$$\delta = \sum_{y \in X} \delta_y \hat{y}.$$

Without loss of generality, we may assume that $\delta_y \in \mathbb{R}^*$ for all $y \in X$.

Lemma 11.1. *Fix a vertex $x \in X$. Write $T \equiv T(x)$, $E_i^* \equiv E_i^*(x)$.*

(i) $T\delta = T\hat{x}$ is an irreducible T-module.

(ii) Given any irreducible T-module W, the following are equivalent:
(iia) $W = T\delta$.
(iib) The diameter $d(W) = d(x)$.
(iic) The endpoint $r(W) = 0$.

Proof. (i) Observe: there exists an irreducible T-module W that contains δ.

Let $V = \sum_i W_i$ be a direct sum decomposition of the standard module. Then

$$\mathrm{Span}(\delta) = E_0 V = \sum_i E_0 W_i.$$

So, $E_0 W_i \neq 0$ for some i. Then,

$$\delta \in E_0 W_i \subseteq W_i.$$

Observe: $T\delta$ is an irreducible T-module.

Since $\delta \in W$, where W is a T-module. As $T\delta \subseteq W$ and W is irreducible, $T\delta = W$.

Observe: $T\delta = T\hat{x}$.

Since $\hat{x} = \delta_x^{-1} E_0^* \delta \in T\delta$, $T\hat{x} \subseteq T\delta$. Since $T\delta$ is irreducible, $T\hat{x} = T\delta$.

(ii) $(a) \to (b)$:

$$E_i^*\delta = \sum_{y \in X, \partial(x,y)=i} \delta_y \hat{y} \neq 0, \quad (0 \le i \le d(x)),$$

because $\delta_y > 0$ for every $y \in X$.

Hence,

$$E_i^* T\delta \neq 0, \quad (0 \le i \le d(x)).$$

Thus, $d(x) = d(W)$.

$(b) \to (c)$: Immediate.

$(c) \to (a)$: Since $r(W) = 0$, $E_0^* W \neq 0$. Hence, $\hat{x} \in W$ and $T\hat{x} \subseteq W$.

By the irreduciblity, we have $T\hat{x} = W$. □

Lemma 11.2. *Assume Γ is bipartite ($X = X^+ \cup X^-$) (X^+ and X^- are nonempty). Then the following are equivalent.*

(i) *There exist α^+ and $\alpha^- \in \mathbb{R}$ such that*

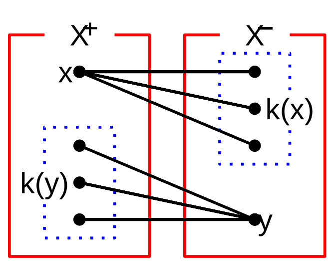
$$\delta_x = \begin{cases} \alpha^+ & \text{if } x \in X^+ \\ \alpha^- & \text{if } x \in X^-. \end{cases}$$

(ii) *There exist k^+ and $k^- \in \mathbb{Z}^{>0}$ such that*

$$k(x) = \begin{cases} k^+ & \text{if } x \in X^+ \\ k^- & \text{if } x \in X^-. \end{cases}$$

In this xase, $k^+k^- = \theta_0^2$, and Γ is called bi-regular.

Proof. $(i) \to (ii)$



$$\begin{aligned}
A\delta &= A\left(\alpha^+ \sum_{x\in X^+} \hat{x} + \alpha^- \sum_{y\in X^-} \hat{y}\right) && (11.1)\\
&= \alpha^+ \sum_{y\in X^-} k(y)\hat{y} + \alpha^- \sum_{x\in X^+} k(x)\hat{x} && (11.2)\\
&= \theta_0\delta. && (11.3)
\end{aligned}$$

So,

$$k(x)\alpha^- = \theta_0\alpha^+, \quad k(y)\alpha^+ = \theta_0\alpha^-.$$

As $\alpha^+ \neq 0$ and $\alpha^- \neq 0$,

$$\begin{aligned}
k^+ &:= k(x) \ \text{ is independent of the choice of } x \in X^+\text{, and} && (11.4)\\
k^- &:= k(y) \ \text{ is independent of the choice of } y \in X^-. && (11.5)
\end{aligned}$$

Moreover, $k^+k^- = \theta_0^2$.

$(ii) \to (i)$ Set

$$\delta' = \sum_{y\in X} \alpha_y \hat{y} \quad \text{where } \alpha = \begin{cases} 1/\sqrt{k^-} & \text{if } y \in X^+ \\ 1/\sqrt{k^+} & \text{if } y \in X^-. \end{cases}$$

Then one checks

$$\begin{aligned}
A\delta' &= A\left(\frac{1}{\sqrt{k^-}} \sum_{y\in X^+} \hat{y} + \frac{1}{\sqrt{k^+}} \sum_{y\in X^-} \hat{y}\right) && (11.6)\\
&= \frac{k^-}{\sqrt{k^-}} \sum_{y\in X^-} \hat{y} + \frac{k^+}{\sqrt{k^+}} \sum_{y\in X^+} \hat{y} && (11.7)\\
&= \sqrt{k^+k^-}\delta' && (11.8)
\end{aligned}$$

Since $\delta' > 0$, $\delta' \in \mathrm{Span}(\delta)$, and $\theta_0 = \sqrt{k^+k^-}$. □

Definition 11.1. For any graph $\Gamma = (X, E)$, fix a vertex $x \in X$. Set $d = d(x)$.

Γ is distance-regular with respect to x, if for all $i : (0 \leq i \leq d)$, and all $y \in X$ such that $\partial(x, y) = i$:

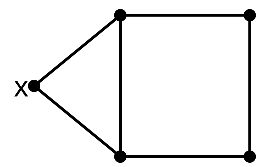
$$\begin{aligned}
c_i(x) &:= |\{z \in X \mid \partial(x,z) = i-1,\ \partial(y,z) = 1\}|, && (11.9)\\
a_i(x) &:= |\{z \in X \mid \partial(x,z) = i,\ \partial(y,z) = 1\}|, && (11.10)\\
b_i(x) &:= |\{z \in X \mid \partial(x,z) = i+1,\ \partial(y,z) = 1\}| && (11.11)
\end{aligned}$$

depends only on i, x, and not on y.

(In this case, $c_0(x) = a_0(x) = b_d(x) = 0$, $c_1(x) = 1$, $b_0(x) = k(x)$ is the valency of x.)

We call $c_i(x)$, $a_i(x)$ and $b_i(x)$ the intersection numbers with respect to x.

Example 11.1.



$$c_0 = 1, \qquad c_1 = 1, \qquad c_2 = 1, \tag{11.12}$$
$$a_0 = 0, \qquad a_1 = 1, \qquad a_2 = 1, \tag{11.13}$$
$$b_0 = 2, \qquad b_1 = 1, \qquad b_2 = 0. \tag{11.14}$$

Chapter 12

Distance-Regular

Monday, February 15, 1993

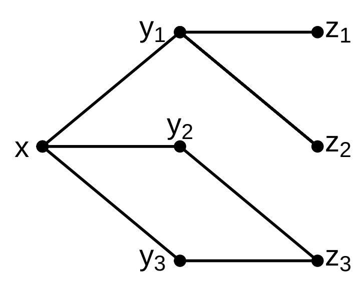
Lemma 12.1. *For any connected graph $\Gamma = (X, E)$, the following are equivalent.*

(i) The trivial $T(x)$-module is thin for all $x \in X$.

(ii) $\left\{ \sum_{y \in X, \partial(x,y)=i} \hat{y} \,|\, 0 \leq i \leq d(x) \right\}$ *is a basis for the trivial $T(x)$-module for every $x \in X$.*

(iii) Γ is distance-regular with respect to x for all $x \in X$.

Note. Let $\Gamma = (X, E)$ be a graph, with $X = \{x, y_1, y_2, y_3, z_1, z_2, z_3\}$, $E = \{xy_1, xy_2, xy_3, y_1z_1, y_1z_2, y_2z_3, y_3z_3\}$.



Then (i), (ii) are not equivalent for a single vertex x.

$$E_0^* T\hat{x} = \langle \hat{x} \rangle, \tag{12.1}$$
$$E_1^* T\hat{x} = \langle \hat{y}_1 + \hat{y}_2 + \hat{y}_3 \rangle, \tag{12.2}$$
$$E_2^* T\hat{x} = \langle \hat{z}_1 + \hat{z}_2 + 2\hat{z}_3 \rangle. \tag{12.3}$$

Proof of Lemma 12.1. $(i) \to (ii)$ Let $\delta = \sum_{y \in X} \delta_y \hat{y}$ be an eigenvector for the maximal eigenvalue θ_0. Then,

$$\sum_{y \in X, \partial(x,y)=1} \hat{y} = A\hat{x} \in T(x)\hat{x} = T(x)\delta \ni E_1^* \delta \tag{12.4}$$

$$= \sum_{y \in X, \partial(x,y)=1} \delta_y \hat{y} \tag{12.5}$$

If the trivial $T(x)$-module is thin,

$$\delta_y = \delta_z \ \text{ for } \ y, z \in X,\ \partial(x,y) = \partial(x,z) = 1.$$

Hence, $\delta_y = \delta_z$ if y and z in X are connected by a path of even length.

So, Γ is regular or bipartite biregular by Lemma 11.2.

In particular, $\delta_y = \delta_z$ if $\partial(x,y) = \partial(x,z)$, as there is a path of length $2 \cdot \partial(x,y)$;

$$y \sim \cdots \sim x \sim \cdots \sim z.$$

Hence,

$$E_i^* \delta \in \mathrm{Span}\left(\sum_{y \in X, \partial(x,y)=i} \hat{y} \right).$$

Since $E_0^*\delta, E_1^*\delta, \ldots, E_d^*\delta$ form a basis for $T(x)\delta$, we have (ii).

$(ii) \to (iii)$ Fix $x \in X$, and let $T \equiv T(x)$, $E_i^* \equiv E_i^*(x)$, and $d \equiv d(x)$.

$$A \sum_{y \in X, \partial(x.y)=i} \hat{y} = \sum_{z \in X} |\{y \in X \mid \partial(y,z) = 1,\ \partial(x,y) = i\}| \hat{z} \tag{12.6}$$

$$= \sum_{z \in X, \partial(x,z)=i-1} b_{i-1}(x,z)\hat{z} \tag{12.7}$$

$$+ \sum_{z \in X, \partial(x,z)=i} a_i(x,z)\hat{z} \tag{12.8}$$

$$+ \sum_{z \in X, \partial(x,z)=i+1} c_{i+1}(x,z)\hat{z} \tag{12.9}$$

$$\in \mathrm{Span}\left\{ \sum_{z \in X, \partial(x,z)=j} \hat{z} \;\middle|\; j = 0, 1, \ldots, d \right\}. \tag{12.10}$$

Hence, $b_{i-1}(x,z)$, $a_i(x,z)$ and $c_{i+1}(x,z)$ depend only on i and x, and not on z. Therefore, Γ is distance-regular with respect to x.

$(iii) \to (i)$ Fix $x \in X$, and let $T \equiv T(x)$, $E_i^* \equiv E_i^*(x)$, and $d \equiv d(x)$. By defintion of distance-regularity, for every i $(0 \le i \le d)$,

$$A\left(\sum_{y\in X,\partial(x,y)=i} \hat{y}\right) = b_{i-1}(x) \sum_{y\in X,\partial(x,y)=i-1} \hat{y} \tag{12.11}$$
$$+ a_i(x) \sum_{y\in X,\partial(x,y)=i} \hat{y} \tag{12.12}$$
$$+ c_{i+1}(x) \sum_{y\in X,\partial(x,y)=i+1} \hat{y}. \tag{12.13}$$

Hence,

$$W = \operatorname{Span}\left\{ \sum_{y\in X,\partial(x,y)=i} \hat{y} \;\middle|\; 0 \le i \le d \right\}$$

is A-invariant and so T-invariant. Since $\hat{x} \in W$, $T\hat{x} = W$ is the trivial module and $T\hat{x}$ is thin. □

Next, we show more is true if $(i) - (iii)$ hold in Lemma 12.1.

In fact, $d(x)$, $a_i(x)$, $c_i(x)$, and $b_i(x)$ are

$$\begin{cases} \text{independent of } X & \text{if } \Gamma \text{ is regular; or} \\ \text{constant over } X^+ \text{ and } X^- & \text{if } \Gamma \text{ is biregular.} \end{cases}$$

Let $\Gamma = (X, E)$ be any (connected) graph. Pick vertices $x, y \in X$.

Let W be a thin, irreducible $T(x)$-module, and measure $m : \mathbb{R} \to \mathbb{R}$ determined by W.

Let W' be a thin, irreducible $T(y)$-module, and measure $m : \mathbb{R} \to \mathbb{R}$ determined by W'.

Recall W, W' are orthogonal if

$$\langle w, w' \rangle = 0 \quad \text{for all } w \in W, w' \in W'.$$

We shall show if W and W' are not orthogonal, then m and m' are related:

$$m \cdot \mathrm{poly}_1 = m' \cdot \mathrm{poly}_2$$

for some polynomials with

$$\deg \mathrm{poly}_1 + \deg \mathrm{poly}_2 \le 2 \cdot \partial(x, y).$$

Notation. V: standard module of Γ.

H: any subspace of V.

$$V = H + H^{\perp} \quad \text{orthogonal direct sum,}$$

and for $v = v_1 + v_2$ $\text{proj}_H : V \to H$ $(v \mapsto v_1)$: linear transformation.

Observe: For every $v \in V$,

$$v - \text{proj}_H v \in H^{\perp}.$$

So,

$$\langle v - \text{proj}_H v, h \rangle = 0 \quad \text{for all} \;\; h \in H \;\text{or,}$$

$$\langle v, h \rangle = \langle \text{proj}_H v, h \rangle \quad \text{for all} \;\; v \in V, \;\; \text{and for all} \;\; h \in H.$$

Theorem 12.1. *Let $\Gamma = (X, E)$ be any graph. Pick vertices $x, y \in X$ and set $\Delta = \partial(x, y)$. Assume*

W: thin irreducible $T(x)$-module with endpoint r, diameter d, and measure m.

W': thin irreducible $T(y)$-module with endpoint r', diameter d', and measure m'.

W and W' are not orghotonal.

Now pick

$$0 \neq w \in E^*_r(x)W, \quad 0 \neq w \in E^*_{r'}(x)W'.$$

Then,

(i) $\text{proj}_{W'} w = p(A)\dfrac{\|w\|}{\|w'\|}w'$

for some $0 \neq p \in \mathbb{C}[\lambda]$ *with* $\deg p \leq \Delta - r' + r, d'$,

$\text{proj}_W w' = p'(A)\dfrac{\|w'\|}{\|w\|}w$

for some $0 \neq p' \in \mathbb{C}[\lambda]$ *with* $\deg p \leq \Delta - r + r', d$.

(ii) For all eigenvalues θ_i of Γ,

$$\frac{\langle E_i w, E_i w' \rangle}{\|w\|\|w'\|} = m(\theta_i)\overline{p'(\theta_i)} = m'(\theta_i)p(\theta_i).$$

(iii) For all eigenvalues θ_i of Γ,

$$p(\theta_i)p'(\theta_i)$$

is a real number in interval $[0, 1]$.

Proof. (i) Since W, W' are not orthogonal, there exist

$$v \in W,\ v' \in W' \ \text{ sich that } \ \langle v, v' \rangle \neq 0.$$

Then there exists $a \in M$ such that

$$v' = aw'.$$

(This is because $w'_i = p'_i(A)w'_0$ and hence for every $v' \in W'$, there is a polynomial $q \in \mathbb{C}[\lambda]$, $q(A)w'_0 = v$.)

We have

$$0 \neq \langle v', v \rangle = \langle aw', v \rangle = \langle w', a^* v \rangle$$

and $a^* v \in W$.

Hence, $\mathrm{proj}_W w' \neq 0$.

Let $p_0, \dots, p_d \in \mathbb{C}[\lambda]$ be from Lemma 9.1.

Then, $w_i = p_i(A)w$ is a basis for $E^*_{r+i}(x)W \quad (0 \leq i \leq d)$.

Hence,

$$\mathrm{proj}_W w' = \alpha_0 w_0 + \cdots + \alpha_d w_d \quad \text{for some } \ \alpha_j \in \mathbb{C}.$$

Set

$$p' := \frac{\|w\|}{\|w'\|} \sum_{i=0}^{d} \alpha_i p_i.$$

Then $0 \neq p' \in \mathbb{C}[\lambda]$ and $\deg p' \leq d$.

Claim: $\alpha_i = 0$ $(\Delta - r + r' < i \leq d)$.

In particular, $\deg p' \leq \Delta - r + r'$.

Pf. Observe:

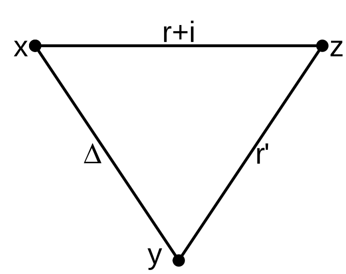
$$w' \in E^*_{r'}(y)V, \quad w \in E^*_r(x)V,$$

for $\partial(x, y) = \Delta$.

$$E^*_{r'}(y)V \cap E^*_{r+i}(x)V = 0$$

by triangle inequality.

$(\Delta = \partial(x, y) < r + i - r'$ or $\Delta + r' < r + i$ by our choice of i.)



Hence,

$$E^*_{r'}(y)V \perp E^*_{r+i}(x)V,$$

or

$$\begin{aligned} 0 &= \langle w', w_i \rangle && (12.14)\\ &= \langle \mathrm{proj}_W w', w_i \rangle && (12.15)\\ &= \sum_{j=0}^{d} \alpha_j \langle w_j, w_i \rangle && (12.16)\\ &= \alpha_i \|w_i\|^2. && (12.17) \end{aligned}$$

Hence, $\alpha_i = 0$. Thus,

$$\begin{aligned} \mathrm{proj}_W w' &= \sum_{i=0}^{\Delta + r' - r} \alpha_i w_i && (12.18)\\ &= \sum_{i=0}^{\Delta + r' - r} \alpha_i p_i(A) w_0 && (12.19)\\ &= p'(A) \frac{\|w'\|}{\|w\|} w. && (12.20) \end{aligned}$$

(ii) We have

$$\begin{aligned} \frac{\langle E_i w, E_i w' \rangle}{\|w\| \|w'\|} &= \frac{\langle E_i w, w' \rangle}{\|w\| \|w'\|} && (12.21)\\ &= \frac{\langle E_i w, \mathrm{proj}_W w' \rangle}{\|w\| \|w'\|} \qquad \text{as } \ \mathrm{proj}_W w' = p'(A) \frac{\|w\|}{\|w'\|} w && (12.22)\\ &= \frac{\langle E_i w, p'(A) w \rangle}{\|w\|^2} && (12.23)\\ &= \frac{\langle E_i w, E_i p'(A) w \rangle}{\|w\|^2} && (12.24)\\ &= \overline{p'(\theta_i)} \frac{\|E_i W\|^2}{\|w\|^2} && (12.25)\\ &= \overline{p'(\theta_i)} m(\theta_i). && (12.26) \end{aligned}$$

Moreover, as $m(\theta_i)$, $m'(\theta_i) \in \mathbb{R}$,

$$\frac{\langle E_i w, E_i w' \rangle}{\|w\| \|w'\|} = \frac{\overline{\langle E_i w, E_i w' \rangle}}{\|w'\| \|w\|} = \overline{\overline{p(\theta_i)} m'(\theta_i)} = p(\theta_i) m'(\theta_i).$$

(iii) Sicne,

$$\frac{|\langle E_i w, E_i w' \rangle \|^2}{\|w\|^2 \|w'\|^2} = p(\theta_i) p'(\theta_i) m(\theta_i) m'(\theta_i),$$

$$p(\theta_i)p'(\theta_i) = \frac{|\langle E_i w, E_i w' \rangle\|^2}{m(\theta_i)m'(\theta_i)\|w\|^2\|w'\|^2} \in \mathbb{R} \tag{12.27}$$

$$= \frac{|\langle E_i w, E_i w' \rangle\|^2}{\frac{\|E_i w\|^2}{\|w\|^2}\frac{\|E_i w'\|^2}{\|w'\|^2}\|w\|^2\|w'\|^2}. \tag{12.28}$$

By Cauchy-Schwartz inequality,

$$(|\langle a, b \rangle| \leq \|a\|\|b\|,)$$

$$\frac{|\langle E_i w, E_i w' \rangle\|^2}{\|E_i w\|^2\|E_i w'\|^2} \leq 1.$$

Hence, we have the assertion. $\square$

Chapter 13

Modules of a DRG

Wednesday, February 17, 1993

Lemma 13.1. *Let $\Gamma = (X, E)$ be any graph. Pick an edge $xy \in E$.*

Assume the trivial $T(x)$-module $T(x)\delta$ is thin with measure m_x,

and the trivial $T(y)$-module $T(y)\delta$ is thin with measure m_y.

Then,

(ia) $\dfrac{m_x(\theta)}{k_x} = \dfrac{m_y(\theta)}{k_y}$ *for all* $\theta \in \mathbb{R} \setminus \{\mathbb{0}\}$.

(ib) $\dfrac{m_x(0)-1}{k_x} = \dfrac{m_y(0)-1}{k_y}$ *for all* $\theta \in \mathbb{R} \setminus \{\mathbb{0}\}$.

$$\Big(\delta = \sum_{y \in X} \delta_y \hat{y} \quad \textit{eigenvector corresponding to the maximal eigenvalue}\Big)$$

Proof. Apply Theorem 12.1,

$$W = T(x)\delta \quad r = 0, \quad d = d(x) \tag{13.1}$$
$$W' = T(y)\delta \quad r' = 0, \quad d' = d(y). \tag{13.2}$$

Take $w = \hat{x}$, $w' = \hat{y}$.

Claim. $\text{proj}_{T(y)\delta}\hat{x} = k_y^{-1} A\hat{y}$.

Pf. Since

$$\hat{y} \in T(y)\delta, \quad A\hat{y} \in T(y)\delta.$$

Show

$$(\hat{x} - {k_y}^{-1} A\hat{y}) \perp (T(y)\delta).$$

Recall

$$A\hat{y} = \sum_{z \in X, yz \in E} \hat{z}.$$

$$\hat{x} - {k_y}^{-1} Ay \in E_1^*(y)V.$$

So,

$$\hat{x} - \frac{1}{k_y} A\hat{y} \perp E_j^*(y)T(y)\delta \quad \text{if } j \neq 1 \ (0 \leq j \leq k(y)).$$

And we have,

$$\left\langle \hat{x} - \frac{1}{k_y} A\hat{y}, A\hat{y} \right\rangle = \left\langle \hat{x}, \sum_{z \in X, yz \in E} \hat{z} \right\rangle - \frac{1}{k_y} \left\| \sum_{z \in X, yz \in E} \hat{z} \right\|^2 \tag{13.3}$$

$$= 1 - 1 \tag{13.4}$$

$$= 0 \tag{13.5}$$

This proves Claim.

Similarly,

$$\operatorname{proj}_{T(x)\delta} \hat{y} = {k_x}^{-1} A\hat{x}.$$

Hence, the polynomials $p, p' \in \mathbb{C}[\lambda]$ from Theorem 12.1 equal

$$\frac{\lambda}{k_y} \quad \text{and} \quad \frac{\lambda}{k_x}$$

respectively.

By Theorem 12.1,

$$\frac{m_x(\theta)\theta}{k_x} = m_x(\theta)\overline{p'(\theta)} = m_y(\theta)\overline{p(\theta)} = \frac{m_y(\theta)\theta}{k_y}.$$

If $\theta \neq 0$, we have (ia).

Also,

$$\frac{1 - m_x(0)}{k_x} = \left(\sum_{\theta \in \mathbb{R} \ \{0\}} m_x(0) \right) \frac{1}{k_x} \qquad \text{by } (ia) \tag{13.6}$$

$$= \left(\sum_{\theta \in \mathbb{R} \ \{0\}} m_y(0) \right) \frac{1}{k_y} \tag{13.7}$$

$$= \frac{1 - m_y(0)}{k_y} \tag{13.8}$$

Hence, we have (ib). $\square$

Theorem 13.1. *Suppose any graph $\Gamma = (X, E)$ is distance-regular with respect to every vertex $x \in X$. (So Γ is regular or biregular by Lemma 12.1.)*

Then,

Case Γ is regular: the diameter $d(x)$ and the intersection numbers $a_i(x)$, $b_i(x)$, $c_i(x)$ $(0 \leq i \leq d(x))$ are independent of $x \in X$.

(And Γ is called distance-regular.)

Case Γ is biregular: $(X = X^+ \cup X^-)$

$d(x)$ and $a_i(x)$, $b_i(x)$, $c_i(x)$ $(0 \leq i \leq d(x))$ are constant over X^+ and X^-. (And Γ is called distance-biregular.)

Proof. We apply Lemma 13.1.

Case Γ: regular.

Then $m_x = m_y$ for all $xy \in E$. Hence, the measure of the trivial $T(x)$-module is independent of $x \in X$.

Case Γ is biregular.

Then $m_x = m_{x'}$ for all $x, x' \in X$ with $\partial(x, x') = 2$.

Hence, the measure of the trivial $T(x)$-module is constant over $x \in X^+$, X^-.

Fix $x \in X$. Write $T \equiv T(x)$, $E_i^* \equiv E_i^*(x)$, $W = T\delta$ with measure m, diameter $d = d(x)$.

We know by Corollary 10.1 that m determines

$$d, \quad a_i(W)\ (0 \leq i \leq d), \quad x_i(W)\ (1 \leq i \leq d)$$

(as $d = D(x) = d(W)$ by Lemma 11.1.)

We shall show that m determines

$$a_i(x),\ c_i(x),\ b_i(x) \quad (0 \leq i \leq d).$$

Observe:

$$a_i(W) = a_i(x) \quad (0 \leq i \leq d) \tag{13.9}$$
$$x_i(W) = b_{i-1}(x)c_i(x) \quad (1 \leq i \leq d) \tag{13.10}$$

HS MEMO

$a_i = a_i(W)$ is an eigenvalue of

$$E_i^* A E_i^* \text{ on } E_i^* W = \langle \sum_{y \in \Gamma_i(x)} \hat{y} \rangle.$$

(See Lemma 12.1.)

$x_i = x_i(W)$ is an eigenvalue of

$$E^*_{i-1}AE^*_iAE^*_{i-1} \text{ on } E^*_{i-1}W,$$

and

$$A \sum_{y\in X,\partial(x,y)} \hat{y} = b_{i-1}(x) \sum_{y\in X,\partial(x,y)=i-1} \hat{y} \tag{13.11}$$
$$+ a_i(x) \sum_{y\in X,\partial(x,y)=i} \hat{y} \tag{13.12}$$
$$+ c_{i+1}(x) \sum_{y\in X,\partial(x,y)=i+1} \hat{y} \tag{13.13}$$

So $x_i = b_{i-1}(x)c_i(x)$.

Set $k^+ = k_x$. Define

$$k^- = \frac{{\theta_0}^2}{k^+},$$

where θ_0 is the maximal eigenvalue. (See Lemma 11.1.)

(So, $k^+ = k^-$ is the valency, if Γ is regular.)

For every i $(0 \le i \le d)$ and for every $z \in X$ with $\partial(x,z) = i$,

$$k_z = c_i(x) + a_i(x) + b_i(x) \tag{13.14}$$
$$= \begin{cases} k^+ & \text{if } i \text{ is even,} \\ k^- & \text{if } i \text{ is odd.} \end{cases} \tag{13.15}$$

Now m determines

$$c_0(x) = a_0(x) = 0, \quad c_1(x) = 1,$$
$$b_0(x) = b_0(x)c_1(x) = x_1(W).$$

$$k^+ = b_0(x) \tag{13.16}$$
$$k^- = {\theta_0}^2/k^+ \tag{13.17}$$
$$c_i(x) = x_i(W)/b_{i-1}(x) \quad (1 \le i \le d) \tag{13.18}$$
$$b_i(x) = \begin{cases} k^+ - a_i(x) - c_i(x) & i\text{; even,} \\ k^- - a_i(x) - c_i(x) & i\text{: odd.} \end{cases} \tag{13.19}$$

This proves the assertions. □

Proposition 13.1. *Under the assumption of Theorem 13.1, the following hold.*

Case Γ*: regular.*

(i) $\dim E_iV = |X|m(\theta_i)$.
(ii) Γ *has exactly* $d+1$ *distinct eigenvalues*

($d = \mathrm{diam}\Gamma = d(x)$, for all $x \in X$).

Case Γ: biregular.

(i) $\dim E_i V = |X^+|m^+(\theta_i) + |X^-|m^-(\theta_i)$.
(ii) Γ has exactly $d^+ + 1$ distinct eigenvalues ($d^+ \geq d^-$).
(iii) If d^+ is odd, the Γ is regular.
(iv) $d^+ = d^-$, or $d^+ = d^- + 1$ is even.
(v) $a_i(x) = 0$ for all i and for all x.

Proof. (i) Suppose Γ is regular.

Let m_x be the measure of the trivial $T(x)$-module,

$$m_x(\theta_i) = \|E_i\hat{x}\|^2, \quad \text{as } \|\hat{x}\| = 1.$$

Now,

$$\begin{aligned} |X|m_x(\theta_i) &= \sum_{x \in X} m_x(\theta_i) && (13.20)\\ &= \sum_{x \in X} \|E_i\hat{x}\|^2 && (13.21)\\ &= \sum_{y,z \in X} |(E_i)_{yz}|^2 && (13.22)\\ &= \mathrm{trace} E_i \overline{E_i}^\top. && (13.23) \end{aligned}$$

Since A is real symmetric and

$$E_i\overline{E_i}^\top = E_i^2 = E_i$$

with E_i symmetric

$$E_i \sim \begin{pmatrix} I & O \\ O & O \end{pmatrix}.$$

$$\mathrm{trace} E_i = \mathrm{rank} E_i = \dim E_i V.$$

Thus, we have the assertion in this case.

Suppose Γ is biregular.

Then, same except,

$$\sum_{x \in X} m_x(\theta_i) = |X^+|m^+(\theta_i) + |X^-|m^-(\theta_i).$$

(ii) Γ: regular. Immediately, if θ is an eigenvalue of Γ, then $m(\theta) \neq 0$.

Γ: biregular. For each $\theta = \theta_i \in \mathbb{R} \ \ \{0\}$,

$$\begin{aligned} m^-(\theta) \neq 0 &\Leftrightarrow m^+(\theta) \neq 0 && (13.24)\\ &\Leftrightarrow \theta \text{ is an eigenvalue of } \Gamma && (13.25)\\ &\left(\frac{m^+(\theta)}{k^+} = \frac{m^-(\theta)}{k^-}\right) && (13.26) \end{aligned}$$

(iv) and (v) are clear.

HS MEMO

(iii) If d^+ is odd, $d^+ = d^-$ and Γ has even number of eigenvalues, i.e., 0 is not an eigenvalue. So A is nonsingular, and Γ is regular.

$\square$

Chapter 14

Parameters of Thin Modules, I

Friday, February 19, 1993

Summary.

Definition 14.1. Assume $\Gamma = (X, E)$ is distance-regular with respect to every vertex $x \in X$.

Notation: Let $x \in X$. The data of the trivial $T(x)$-module.

	Case DR	Case DBR
valencyk_x	k	$\begin{cases} k^+ & \text{if } x \in X^+ \\ k^- & \text{if } x \in X^- \end{cases}$
x-diameter D_x	D	$\begin{cases} D^+ & \text{if } x \in X^+ \\ D^- & \text{if } x \in X^- \end{cases}$
measure m_x	m	$\begin{cases} m^+ & \text{if } x \in X^+ \\ m^- & \text{if } x \in X^- \end{cases}$
int. number $c_i(x)$	c_i	$\begin{cases} c_i^+ & \text{if } x \in X^+ \\ c_i^- & \text{if } x \in X^- \end{cases}$
int. number $b_i(x)$	b_i	$\begin{cases} b_i^+ & \text{if } x \in X^+ \\ b_i^- & \text{if } x \in X^- \end{cases}$
int. number $a_i(x)$	a_i	0

Call m, $m^{\pm 1}$ the measure of Γ.

Assume $\Gamma = (X, E)$ is distance-regular.

To what extent do a_i's, b_i's and c_i's determine the structure of irreducible $T(x)$-modules? In general, the following hold.

Lemma 14.1. *Assume $\Gamma = (X, E)$ is distance-regular. Pick $x \in X$. Let W be a thin irreducible $T(x)$-module with endpoint r, diameter d and measure m_W.*

(i) There is a unique polynomial $f_W \in \mathbb{C}[\lambda]$ with the following properties.

(ia) $\deg f_W \leq D$ (diameter of Γ).

(ib) $m_W(\theta) = m(\theta) f_W(\theta)$ for every $\theta \in \mathbb{R}$, where m is the measure of Γ.

Moreover, $f_W \in \mathbb{R}[\lambda]$, and

(ii) $\deg f_W \leq 2r$.
(iii) For all eigenvalues θ_i of Γ, $\lambda - \theta_i$ is a factor of f_W whenever, $E_i W = 0$.

In particular, $2r - D + d \geq 0$.

Proof. Let $\theta_0, \dots, \theta_D$ denote distinct eigenvalues of Γ. Then $m(\theta_i) \neq 0$ $(0 \leq i \leq D)$ by Proposition 13.1.

There exists a unique $f_W \in \mathbb{C}[\lambda]$ with $\deg f_W \leq D$ such that

$$f_W(\theta_i) = \frac{m_W(\theta_i)}{m(\theta_i)} \quad (0 \leq i \leq D)$$

by polynomial interpolation.

$f_W \in \mathbb{R}[\lambda]$ since

$$\theta_0, \dots, \theta_D \in \mathbb{R} \quad \text{and} \quad f_W(\theta_0), \dots, f_W(\theta_D) \in \mathbb{R}.$$

(ii) Without loss of generality, we may assume $r < D/2$, else trivial.

Pick $0 \neq w \in E_r^*(x)W$.

$$w = \sum_{y \in W, \partial(x,y) = r} \alpha_y \hat{y} \quad \text{for some } \alpha_y \in \mathbb{C}.$$

Pick $y \in X$ such that $\alpha_y \neq 0$.

Set W' be the trivial $T(y)$-module. ($\langle w, \hat{y} \rangle \neq 0$, as $W \not\perp W'$.)

$$r' = 0, \quad m' = m, \quad \Delta = r.$$

Apply Theorem 12.1, we have

$$\deg p \leq \Delta - r' + r = 2r, \quad p \neq 0 \tag{14.1}$$
$$\deg p' \leq \Delta - r + r' = 0, \quad p' \neq 0. \tag{14.2}$$

$$m_W(\theta)\overline{p'(\theta)} = m(\theta) p(\theta) \quad (\text{ for all } \theta \in \mathbb{R}).$$

So,

$$\deg p/\bar{p}' \leq 2r,$$

and $p/\bar{p}'$ satisfies the conditions of f_W.

$$\left(\frac{p(\theta)}{\bar{p}'(\theta)} = \frac{m_W(\theta)}{m(\theta)}\right)$$

(iii)

$$E_iW = 0 \Rightarrow m_W(\theta_i) = 0 \Rightarrow f_W(\theta_i) = 0.$$

that is, $E_iW = 0$. Hence θ_i is a root of $f_W(\lambda) = 0$. So,

$$2r \geq \deg f_W \geq |\{\theta_i \mid E_iW = 0\}| = D - d.$$

Hence,

$$2r - D + d \geq 0.$$

This proves the assertions. □

Lemma 14.2. *Let $\Gamma = (X, E)$ be any distance-regular graph with valency k, diameter D ($D \geq 2$), measure m, and eigenvalues*

$$k = \theta_0 > \theta_1 > \cdots > \theta_D.$$

Pick $x \in X$. Let W be a thin irreducible $T(x)$-module with endpoint $r = 1$, diameter d and measure $m_W = mf_W$. Then one fo the following cases $(i) - (iv)$ occurs.

Case	d	$f_W(\lambda)$	$a_0(W)$
(i)	$D-2$	$\frac{(\lambda-k)(\lambda-\theta_1)}{k(\theta_1+1)}$	$-\frac{b_1}{\theta_1+1} - 1$
(ii)	$D-2$	$\frac{(\lambda-k)(\lambda-\theta_D)}{k(\theta_D+1)}$	$-\frac{b_1}{\theta_D+1} - 1$
(iii)	$D-1$	$\frac{k-\lambda}{k}$	-1
(iv)	$D-1$	$\frac{(\lambda-k)(\lambda-\beta)}{k(\beta+1)}$	$-\frac{b_1}{\beta+1} - 1$

for some $\beta \in \mathbb{R}$ with $\beta \in (-\infty, \theta_D) \cup (\theta_1, \infty)$. Moreover, the isomorphism class of W is determined by $a_0(W)$.

Note. By (iii), the possible "shapes" of a thin irreducible $T(x)$-modules are:

$$r = 0 \quad d = D, \tag{14.3}$$
$$r = 1 \quad d = D - 1, \tag{14.4}$$
$$r = 1 \quad d = D - 2. \tag{14.5}$$

Chapter 15

Parameters of Thin Modules, II

Monday, February 22, 1993

Proof of Lemma 14.2 Continued.

We have $\deg f_W \leq 2$ by Lemma 14.1 (ii).

Also by Lemma 11.1, $E_0 W = 0$.

(As otherwise $\langle \delta \rangle = E_0 V \subseteq W$ and $r(W) = 0$.)

Hence, $\lambda - \theta_0 = \lambda - k$ is a factor of f_W by Lemma 14.1 (iii).

Let $p_0, p_1, \ldots, p_D$ denote the polynomials for the trivial $T(x)$-module from Lemma 9.1.

Recall,

$$\sum_{\theta \in \mathbb{R}} m(\theta) p_i(\theta) p_j(\theta) = \delta_{ij} x_1 x_2 \cdots x_i \quad (0 \leq i, j \leq D) \tag{15.1}$$

$$= \delta_{ij} b_0 b_1 \cdots b_{i-1} c_1 c_2 \cdots c_i. \tag{15.2}$$

Note that $x_i = b_{i-1} c_i$ is in the proof of Theorem 13.1.

By construction,

$$p_0(\lambda) = 1, \tag{15.3}$$

$$p_1(\lambda) = \lambda, \tag{15.4}$$

$$p_2(\lambda) = \lambda^2 - a_1 \lambda - k. \tag{15.5}$$

Apparently,

$$f_W = \sigma_0 p_0 + \sigma_1 p_1 + \sigma_2 p_2$$

for some $\sigma_0, \sigma_1, \sigma_2 \in \mathbb{C}$.

Claim:

$$\sigma_0 = 1, \tag{15.6}$$
$$\sigma_1 = \frac{a_0(W)}{k}, \tag{15.7}$$
$$\sigma_2 = -\frac{1 + a_0(W)}{kb_1}. \tag{15.8}$$

Pf of Claim.

$$1 = \sum_{\theta \in \mathbb{R}} m_W(\theta) \tag{15.9}$$
$$= \sum_{\theta \in \mathbb{R}} m(\theta) f_W(\theta) \tag{15.10}$$
$$= \sum_{j=0}^{2} \sigma_j \left(\sum_{\theta \in \mathbb{R}} m(\theta) p_j(\theta) \right) \tag{15.11}$$
$$= \sigma_0. \tag{15.12}$$

We applied Lemma 10.1 (*ib*), Lemma 14.1 (*ib*), and Lemma 10.1 (*i*) in this order. Next by Lemma 10.1 (*ii*), and $p_1(\theta) = \theta$,

$$a_0(W) = \sum_{\theta \in \mathbb{R}} m_W(\theta)\theta \tag{15.13}$$
$$= \sum_{\theta \in \mathbb{R}} m(\theta) f_W(\theta)\theta \tag{15.14}$$
$$= \sum_{j=0}^{2} \sigma_j \sum_{\theta \in \mathbb{R}} m(\theta) p_j(\theta) p_1(\theta) \tag{15.15}$$
$$= \sigma_1 x_1(T\delta) \tag{15.16}$$
$$= \sigma_1 b_0 c_1 \tag{15.17}$$
$$= \sigma_1 k. \tag{15.18}$$

So far,

$$f_W(\lambda) = 1 + \frac{a_0(W)}{k}\lambda + \sigma_2(\lambda^2 - a_1\lambda - k).$$

But,

$$0 = f_W(k) \tag{15.19}$$
$$= 1 + a_0(W) + \sigma_2 k(k - a_1 - 1) \tag{15.20}$$
$$= 1 + a_0(W) + \sigma_2 k b_1. \tag{15.21}$$

Thus,

$$\sigma_2 = -\frac{1 + a_0(W)}{kb_1}.$$

This proves Claim.

Case: $a_0(W) = -1$.

Here, $\sigma_2 = 0$ and

$$f_W(\lambda) = 1 + \frac{a_0(W)\lambda}{k} = 1 - \frac{\lambda}{k}.$$

Also,

$$d + 1 = |\{\theta \mid \theta \text{ is an eigenvalue of } \Gamma,\ f_W(\theta) \neq 0\}| = D.$$

Case: $a_0(W) \neq -1$.

Here, $\sigma_2 \neq 0$, and $\deg f_W = 2$. So,

$$f_W(\lambda) = (\lambda - k)(\lambda - \beta)\alpha$$

for some $\alpha, \beta \in \mathbb{C},\ \alpha \neq 0$.

Comparing the coefficients in

$$(\lambda - k)(\lambda - \beta)\alpha = 1 + \frac{a_0(W)}{k}\lambda - \frac{a_0(W)+1}{kb_1}(\lambda^2 - a_1\lambda - k),$$

we find

$$\alpha = -\frac{a_0(W)+1}{kb_1}, \tag{15.22}$$

$$-(k+\beta)\alpha = \frac{a_0(W)}{k} + \frac{a_0(W)+1}{kb_1}a_1, \tag{15.23}$$

$$k\beta\alpha = 1 + \frac{a_0(W)+1}{b_1}. \tag{15.24}$$

Hence,

$$-\beta(a_0(W)+1) = b_1 + (a_0(W)+1).$$

Thus, we have

$$(1 + a_0(W))(1+\beta) = -b_1. \tag{15.25}$$

In particular, $\beta \neq -1$, and

$$\alpha = -\frac{1 + a_0(W)}{kb_1} = \frac{1}{k(\beta+1)}.$$

Also, by Definition 9.2,

$$0 \leq m_W(\theta) \tag{15.26}$$

$$= m(\theta) f_W(\theta) \quad (\text{for all } \theta \in \mathbb{R}). \tag{15.27}$$

But if θ is an eigenvalue of Γ,

$$0 < m(\theta).$$

So,

$$0 \le f_W(\theta) \tag{15.28}$$
$$= \frac{(\theta-k)(\theta-\beta)}{k(\beta+1)}. \tag{15.29}$$

Either

$$\beta+1>0 \to \theta-\beta \le 0 \ \text{ or } \ \beta \ge \theta_1,$$

or

$$\beta+1<0 \to \theta-\beta \ge 0 \ \text{ or } \ \beta \le \theta_D.$$

If $\beta=\theta_1$,

$$a_0(W) = -\frac{b_1}{\beta+1} - 1 = -\frac{b_1}{\theta_1+1} - 1 \tag{15.30}$$
$$f_W(\lambda) = \frac{(\lambda-k)(\lambda-\theta_1)}{k(\theta_1+1)}, \tag{15.31}$$

and we have (i).

If $\beta=\theta_D$,

$$a_0(W) = -\frac{b_1}{\theta_D+1} - 1 \tag{15.32}$$
$$f_W(\lambda) = \frac{(\lambda-k)(\lambda-\theta_D)}{k(\theta_D+1)}, \tag{15.33}$$

and we have (ii).

If $\beta \notin \{\theta_1, \theta_D\}$,

$$\beta \in (-\infty, \theta_D) \cup (\theta_1, \infty),$$

we have (iv).

Note using (15.25), we have (iv).

$\square$

Note. Using (15.25),

$$a_0(W) \to \beta \to f_W \to m_W \to \text{isomorphism class of } W.$$

Note on Lemma 14.2. In fact, $\theta_1 > -1$, $\theta_D < -1$ if $D \ge 2$.

Definition 15.1. The complete graph K_n has n vertices and diameter $D=1$, i.e., $xy \in E$ for all vertices x, t.

K_n is distance-regular with valency $k = n-1$ and $a_1 = n-2$, $D = 1$. Moreover, it has two distince eigenvalues θ_0, θ_1.

Recall, $\theta_0, \dots, \theta_D$ are roots of p_{D+1}, i.e., $D+1$ st polynomial for the trivial module.

$$
\begin{aligned}
p_0 &= 1, && (15.34)\\
p_1 &= \lambda, && (15.35)\\
p_2 &= \lambda^2 - a_1\lambda - k && (15.36)\\
&= \lambda^2 - (n-2)\lambda - (n-1) && (15.37)\\
&= (\lambda - (n-1))(\lambda + 1). && (15.38)
\end{aligned}
$$

The roots are $\theta_0 = n - 1 = k$ and $\theta_1 = -1$.

Lemma 15.1. *Let $\Gamma = (X, E)$ be distance-regular of diameter $D \geq 1$ with distinct eigenvalues*

$$k = \theta_0 > \theta_1 > \cdots > \theta_D.$$

(i) $\theta_D \leq -1$ with equality if and only if $D = 1$.

(ii) $\theta_1 \geq -1$ with equality if and only if $D = 1$.

Proof. (i) Suppose $\theta_D \geq -1$.

Then $I + A$ is positive semi-definite.

By Lemma 2.1, there exists vectors $\{v_x \mid x \in X\}$ in a Euclidean space such that

$$
\begin{aligned}
\langle v_x, v_y\rangle &= (I + A)_{xy} && (15.39)\\
&= \begin{cases} 1 & \text{if } x = y \text{ or } xy \in E,\\ 0 & \text{othewise.}\end{cases} && (15.40)
\end{aligned}
$$

For every $xy \in E$,

$$\langle v_x, v_y\rangle = \|v_x\|\|v_y\| = 1.$$

Hence, $v_x = v_y$, and v_x is independent of $x \in X$.

Thus $\langle v_x, v_y\rangle = 1$ for all $x, y \in X$.

We have $I + A = J$, (all 1's matrix), and $D = 1$.

(ii) Let m be the trivial measure. Then,

$$
\begin{aligned}
1 &= \sum_{\theta\in\mathbb{R}} m(\theta) + \sum_{\theta\in\mathbb{R}} m(\theta)\theta && (15.41)\\
&= \sum_{\theta\in\mathbb{R}} m(\theta)(\theta + 1) && (15.42)\\
&= m(k)(k+1) + \sum_{\theta\neq k} m(\theta)(\theta+1) && (15.43)\\
&\leq (k+1)|X|^{-1}. && (15.44)
\end{aligned}
$$

Note that $m(k) = |X|^{-1} \dim E_0 V = |X|^{-1}$.

So $k+1 \geq |X|$ or $k = |X|-1$. Thus, $xy \in E$ for every $x, y \in X$, and $D = 1$. $\square$

Note. Lemma 15.1 does not require distance-regular assumption.

Chapter 16

Thin Modoles of a DRG

Wednesday, February 24, 1993

Let $\Gamma = (X, E)$ denote any graph of diameter D.

Definition 16.1. For all integers i, the i-th incidence matrix $A_i \in \mathrm{Mat}_X(\mathbb{C})$ satisfies

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x,y) = i, \\ 0 & \text{if } \partial(x,y) \neq i, \end{cases} \quad (x, y \in X).$$

Observe,

$$\begin{aligned} A_0 &= I & &\text{(identity)} & &(16.1) \\ A_1 &= A & &\text{(adjacency matrix)} & &(16.2) \\ A_0 + A_1 + \cdots + A_D &= J & &\text{(all 1's matrix).} & &(16.3) \end{aligned}$$

In general, A_i may not belong to Bose-Mesner algebra.

Lemma 16.1. *Assume $\Gamma = (X, E)$ is distance-regular with diameter $D \geq 1$ and intersection numbers c_i, a_i, b_i.*

(i)

$$AA_i = c_{i+1}A_{i+1} + a_iA_i + b_{i-1}A_{i-1}, \quad (0 \leq i \leq D,\ A_{-1} = A_{D+1} = O).$$

(ii) $A_i = \dfrac{p_i(A)}{c_1c_2\cdots c_i}$, $\quad (0 \leq i \leq D)$*, where $p_0, p_1, \ldots, p_D$ are polynomials for the trivial module from Lemma 9.1.*

(iii) $A_0, A_1, \ldots, A_D$ form a bais for Bose-Mesner algebra M.

(iv) For all distances h, i, j $\quad (0 \leq, i, j, h \leq D)$, and for all vertices $x, y \in X$ with $\partial(x, y) = h$, the constant

$$p^h_{i,j} = |\{z \in X \mid \partial(x,z) = i,\ \partial(y,z) = j\}|$$

depends only on h, i, j and not on x, y.

(v) $E_0 = \dfrac{1}{|X|}J$.

Proof.

(i) Pick $x \in X$. Apply each side to $\hat{x}$, we want to show that

$$AA_i\hat{x} = c_{i+1}A_{i+1}\hat{x} + a_iA_i\hat{x} + b_{i-1}A_{i-1}\hat{x}.$$

$$\begin{aligned}
\text{LHS} &= A\left(\sum_{y\in X, \partial(x,y)=i} \hat{y}\right) && (16.4)\\
&= c_{i+1}\left(\sum_{z\in X, \partial(x,z)=i+1} \hat{z}\right) + a_i\left(\sum_{z\in X, \partial(x,z)=i} \hat{z}\right) + b_{i-1}\left(\sum_{z\in X, \partial(x,z)=i-1} \hat{z}\right) && (16.5)\\
&= \text{RHS}. && (16.6)
\end{aligned}$$

(ii) Recall (Lemma 9.1)

$$Ap_i(A) = p_{i+1}(A) + a_ip_i(A) + b_{i-1}c_ip_{i-1}(A) \quad (0 \le i \le D).$$

Dividing by $c_1c_2\cdots c_i$, we have

$$A\frac{p_i(A)}{c_1c_2\cdots c_i} = c_{i+1}\frac{p_{i+1}(A)}{c_1c_2\cdots c_{i+1}} + a_i\frac{p_i(A)}{c_1c_2\cdots c_i} + b_{i-1}\frac{p_{i-1}(A)}{c_1c_2\cdots c_i}.$$

So, A_i, $p_i(A)/(c_1c_2\cdots c_i)$ satisfy the same recurrence.

Also boundary condition,

$$A_0 = p_0(A) = I.$$

Hence,

$$A_i = \frac{p_i(A)}{c_1c_2\cdots c_i} \quad (0 \le i \le D).$$

(iii) Since $E_0, E_1, \ldots, E_D$ form a basis for M, $\dim M = D + 1$.

Observe $A_0, A_1, \ldots, A_D \in M$ by (ii), $A_0, A_1, \ldots, A_D$ are linearly independent, since $p_0, p_1, \ldots, p_D$ are linearly independent.

Thus, $A_0, A_1, \ldots, A_D$ form a basis for M.

(iv) $A_0, A_1, \ldots, A_D$ form a basis for an algebra M,

$$A_iA_j = \sum_{\ell=0}^{D} p_{ij}^{\ell} A_\ell \quad \text{for some } \ p_{ij}^{\ell} \in \mathbb{C}. \tag{16.7}$$

Fix $h \quad (0 \leq h \leq D)$. Pick $x, y \in X$ with $\partial(x, y) = h$.

Compute x, y entry in (16.7),

$$(A_iA_j)_{xy} = \sum_{z \in X} (A_i)_{xz}(A_j)_{zy} \tag{16.8}$$

$$= \sum_{z \in X, \partial(x,z)=i, \partial(y,z)=j} 1 \cdot 1 \tag{16.9}$$

$$= |\{z \in X \mid \partial(x, z) = i, \partial(y, z) = j\}|. \tag{16.10}$$

On the other hand,

$$\left(\sum_{\ell=0}^{D} p_{ij}^{\ell} A_\ell \right)_{xy} = p_{ij}^{h}(A_h)_{xy} = p_{ij}^{h}.$$

(v) $\frac{1}{|X|}J$ is the orthogonal projection onto $\text{Span}(\delta) = E_0V$. Hence,

$$\frac{1}{|X|} = E_0.$$

This proves the assertions.

□

Theorem 16.1. *Let $\Gamma = (X, E)$ be distance-regular with diameter $D \geq 2$ and intersection numbers c_i, a_i, b_i. Pick a vertex $x \in X$. Let W be a thin irreducible $T(x)$-module with endpoint $r = 1$ and diameter d ($d = D - 2$ or $D - 1$). Set $\gamma_0 = a_0(W) + 1$.*

(i) The scalars

$$\gamma_i := \frac{c_2c_3 \cdots c_{i+1}b_2b_3 \cdots b_{i+1}\gamma_0}{x_1(W)x_2(W) \cdots x_i(W)} \quad (0 \leq i \leq d) \tag{16.11}$$

$a_i(W), x_i(W)$ are algebraic integers in $\mathbb{Q}[\gamma_0]$. In particular, if $\gamma_0 \in \mathbb{Q}$, then γ_i, $a_i(W)$ and $x_i(W)$ are integers for all i.

(ii) The numbers, $\gamma_i, a_i(W), x_i(W)$ can all be determined from γ_0 and the intersection numbers of Γ in order

$$x_1(W), \gamma_1, a_1(W), x_2(W), \gamma_2, a_2(W), \ldots$$

using (i),

$$x_i(W) = c_ib_i + \gamma_{i-1}(a_i + c_i - c_{i+1} - a_{i-1}(W)) \quad (1 \leq i \leq D - 1), \tag{16.12}$$

and

$$a_i(W) = \gamma_i - \gamma_{i-1} + a_i + c_i - c_{i+1} \quad (1 \le i \le D). \tag{16.13}$$

Note.

$$p_i = p_1^W + \gamma_{i-1} p_{i-1}^W - c_i(p_{i-1}^W + \gamma_{i-2} p_{i-2}^W),\ (\gamma_{-1} = -\gamma_{-2} = 0,\ 0 \le i \le d+1).$$

Proof. Set

$$\tilde{A}_i = A_0 + A_1 + \cdots + A_i \quad (0 \le i \le D).$$

Claim 1. $A\tilde{A}_i = c_{i+1}\tilde{A}_{i+1} + (a_i - c_{i+1} + c_i)\tilde{A}_i + b_i\tilde{A}_{i-1} \quad (0 \le i \le D-1)$.

Proof of Claim 1.

$$\begin{aligned}
\text{LHS} &= \sum_{j=0}^{i} AA_j && (16.14)\\
&= \sum_{j=0}^{i} (c_{j+1}A_{j+1} + a_jA_j + b_{j-1}A_{j-1}) && (16.15)\\
&= \sum_{j=0}^{i-1} A_j(c_j + a_j + b_j) + A_i(c_i + a_i) + A_{i+1}c_{i+1} && (16.16)\\
&= k(A_0 + \cdots + A_{i-1}) + (a_i + c_i)A_i + c_{i+1}A_{i+1}. && (16.17)
\end{aligned}$$

$$\begin{aligned}
\text{RHS} = {} & c_{i+1}(A_0 + A_1 + \cdots + A_{i-1} + A_i + A_{i+1}) && (16.18)\\
& + (a_i - c_{i+1} + c_i)(A_0 + A_1 + \cdots + A_{i-1} + A_i) && (16.19)\\
& + b_i(A_0 + A_1 + \cdots + A_{i-1}) && (16.20)\\
= {} & k(A_0 + \cdots + A_{i-1}) + A_i(a_i + c_i) + A_{i+1}c_{i+1}. && (16.21)
\end{aligned}$$

This proves Claim 1.

Now pick $0 \ne w \in E_1^*(x)W$ and let

$$w = \sum_{z \in X, \partial(x,z)=1} \alpha_z \hat{z}.$$

Pick y, where $\alpha_y \ne 0$.

For all i $(0 \le i \le D)$, define

$$\begin{aligned}
B_i &= \tilde{A}_i(\hat{x} - \hat{y}) && (16.22)\\
&= \sum_{z \in X, \partial(x,z) \le i} \hat{z} - \sum_{z \in X, \partial(y,z) \le i} \hat{z} && (16.23)\\
&= \sum_{z \in X, \partial(x,z)=i, \partial(y,z)=i+1} \hat{z} - \sum_{z \in X, \partial(y,z)=i+1, \partial(y,z)=i} \hat{z}. && (16.24)
\end{aligned}$$

Note that $B_D = O$, $B_0 = \hat{x} - \hat{y}$, and

$$\langle B_0, w_0 \rangle = -\alpha_y \neq 0.$$

From Claim 1,

$$AB_i = c_{i+1}B_{i+1} + (a_i - c_{i+1} + c_i)B_i + b_i B_{i-1} \ (0 \leq i \leq D), \ B_{-1} = O.$$

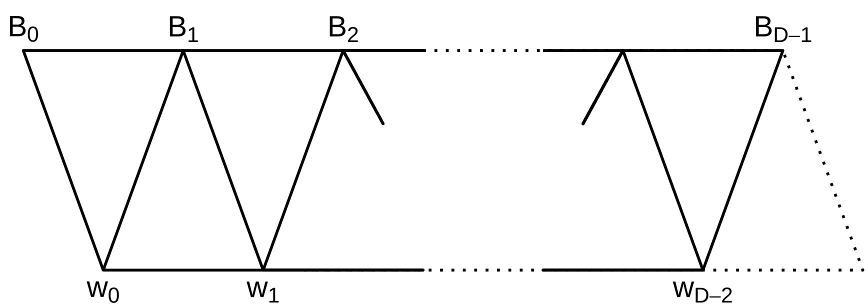
Let $p_0^W, \dots, p_d^W$ denote polynomials for W from Lemma 9.1. So,

$$w_i = p_i^W(A)w \in E^*_{1+i}(x)W, \quad (0 \leq i \leq d).$$

Claim 2. $\langle w_i, B_j \rangle = 0$ if $j \notin \{i, i+1\}$, $(0 \leq i \leq d, 0 \leq j \leq D)$.

Proof of Claim 2.

$$w_i \in E^*_{1+i}(x)W, \quad B_j \in E^*_j(x)W + E^*_{j+1}(x)W.$$



Vertical lines indicate possible non-orthogonality.

Compute

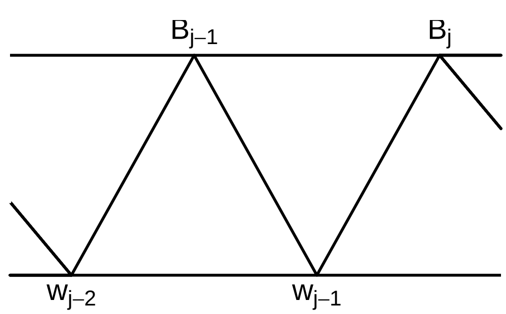
$$\langle Aw_i, B_j \rangle = \langle w_i, AB_j \rangle, \quad (0 \leq i \leq D, \ 0 \leq j \leq D-1). \tag{16.25}$$

$$\text{LHS} = \langle w_{i+1}, B_j \rangle + a_i(W)\langle w_i, B_j \rangle + x_i(W)\langle w_{i-1}, B_j \rangle \tag{16.26}$$

$$\text{RHD} = b_j \langle w_i, B_{j-1} \rangle + (a_j - c_{j+1} + c_j)\langle w_i, B_j \rangle + c_{j+1}\langle w_i, B_{j+1} \rangle. \tag{16.27}$$

Evaluate for $i = j-2, j-1, j, j+1$.

Set $i = j-2$.



Then (16.25) becomes

$$\langle w_{j-1}, B_j\rangle = b_j\langle w_{j-2}, B_{j-1}\rangle \quad (2 \le j \le D-1).$$

By induction,

$$\langle w_{j-1}, B_j\rangle = b_2 b_3 \cdots b_j \langle w_0, B_1\rangle \quad (1 \le j \le D-1).$$

Define

$$\gamma_0 = \frac{\langle w_0, B_1\rangle}{\langle w_0, B_0\rangle}.$$

(We will show $\gamma_0 = 1 + a_0(W)$.)

Then,

$$\langle w_{j-1}, B_j\rangle = b_2 b_3 \cdots b_j \gamma_0 \langle w_0, B_0\rangle. \tag{16.28}$$

Set $i = j+1$. Then (16.25) becomes

$$x_{j+1}(W)\langle w_j, B_j\rangle = c_{j+1}\langle w_0, B_{j+1}\rangle \quad (0 \le j \le d).$$

Hence,

$$\langle w_j, B_j\rangle = \frac{x_1(W)\cdots w_j(W)}{c_1 c_2 \cdots c_j}\langle w_0, B_0\rangle \quad (0 \le j \le d). \tag{16.29}$$

Set $i = j-1$. Then (16.25) becomes

$$\langle w_j, B_j\rangle + a_{j-1}(W)\langle w_{j-1}, B_j\rangle = (a_j - c_{j+1} + c_j)\langle w_{j-1}, B_j\rangle + b_j\langle w_{j-1}, B_{j-1}\rangle.$$

Evaluate this using (16.28) and (16.29). ($\langle w_0, B_0\rangle \neq 0$). Then we have

$$\frac{w_1(W)\cdots x_j(W)}{c_1\cdots c_j} + (a_{j-1}(W) - a_j + c_{j+1} - c_j)b_2\cdots b_j\gamma_0 = b_j\frac{x_1(W)\cdots x_{j-1}(W)}{c_1\cdots c_{j-1}},$$

$$\left(\gamma_i := \frac{c_2 c_3 \cdots c_{i+1} b_2 b_3 \cdots b_{i+1}\gamma_0}{x_1(W)x_2(W)\cdots x_i(W)}\right).$$

$$\frac{x_j(W)}{c_j} = b_j + \frac{c_1 c_3\cdots c_{j-1} b_2 b_3 \cdots b_j \gamma_0}{x_1(W)x_2(W)\cdots x_{j-1}(W)}(a_j + c_j - c_{j+1} - a_{j-1}).$$

So,

$$x_j(W) = c_j b_j + \gamma_{j-1}(a_j + c_j - c_{j+1} - a_{j-1}(W)).$$

This proves (16.12).

Set $i = j$. Then (16.25) becomes

$$a_j(W)\langle w_j, B_j\rangle + x_j(W)\langle w_{j-1}, B_j\rangle = (a_j - c_{j+1} + c_j)\langle w_j, B_j\rangle + c_{j+1}\langle w_j, B_{j+1}\rangle.$$

$$(a_j(W) - (a_j - c_{j+1} + c_j))\frac{x_1(W)\cdots x_j(W)}{c_1\cdots c_j} x_j(W) b_2\cdots b_j\gamma_0 - c_{j+1}b_2\cdots b_{j+1}\gamma_0 = 0.$$

Thus,

$$a_j(W) - (a_j - c_{j+1} + c_j) + \frac{c_1 \cdots c_j b_2 \cdots b_j \gamma_0}{x_1(W) \cdots x_{j-1}(W)} - \frac{c_1 \cdots c_j c_{j+1} b_2 \cdots b_{j+1} \gamma_0}{x_1(W) \cdots x_j(W)} = 0,$$

or

$$a_j(W) = a_j + c_j - c_{j+1} - \gamma_{j-1} + \gamma_j.$$

This proves (16.13).

Also by setting $i = j = 0$, we have

$$\begin{aligned} a_0(W)\langle w_0, B_0\rangle &= (a_0 - c_1 + c_0)\langle w_0, B_0\rangle + c_1\langle w_0, B_1\rangle && (16.30) \\ &= -\langle w_0, B_0\rangle + \gamma_0\langle w_0, B_0\rangle. && (16.31) \end{aligned}$$

Hence,

$$\gamma_0 = 1 + a_0(W).$$

Both $a_i(W)$ and $x_i(W)$ are algebraic integers, since they are eigenvalues of matrices with integer entries, namely,

$$E^*_{i+1}(x)AE^*_{i+1}(x) \text{ and } E^*_i(x)AE^*_{i+1}(x)AE^*_i(x).$$

Also $\gamma_0 = 1 + a_0(W)$ is an algebraic integer, and $\gamma_i - \gamma_{i-1}$ is an algebraic integer by (16.12).

Hence, γ_i is an algebraic integer by induction.

This completes the proof of Theorem 16.1. □

Example 16.1 (D=2)**.**

$$D = 2 \Leftrightarrow \text{strongly regular.}$$

Free parameters are k, a_1, c_2. Let W be an irreducible module of endpoint 1. The matrix representation of $A|_W$ is

$$\begin{pmatrix} a_0(W) & x_1(W) \\ 1 & a_1(W) \end{pmatrix}.$$

$a_0(W)$: free.

$$\begin{aligned} x_1(W) &= c_1 b_1 + (a_0(W) + 1)(a_1 + c_1 - c_2 - a_0(W)) && (16.32) \\ &= k - a_1 - 1 + a_1 a_0(W) + a_0(W) - c_2 a_0(W) - a_0(W)^2 && (16.33) \\ &\quad + a_1 + a - c_2 - a_0(W) && (16.34) \\ &= a_1 a_0(W) - c_2 a_0(W) + k - c_2 - a_0(W)^2, && (16.35) \\ \gamma_1 &= 0, && (16.36) \\ a_1(W) &= -(a_0(W) + 1) + a_1 + c_1 - c_2 && (16.37) \\ &= -a_0(W) + a_1 - c_2. && (16.38) \end{aligned}$$

Then the matrix has eigenvalues θ, θ_1. There is one feasible condition: $a_0(W)$ is an algebraic integer.

Example 16.2 (D=3)**.** Free parameters c_2, c_3, k, a_1, a_2. The matrix representation becomes

$$A|_W = \begin{pmatrix} a_0(W) & x_1(W) & 0 \\ 1 & a_1(W) & x_2(W) \\ 0 & 1 & a_2(W) \end{pmatrix}.$$

Here, $a_0(W)$ is free $(= \gamma - 1)$

$$x_1(W) = k - 1 - a_1 + \gamma_0(a_1 + 1 - c_2 - a_0(W)) \tag{16.39}$$
$$= \gamma_0(a_1 - c_2 - a_0(W)) + k - a_1 + a_0(W). \tag{16.40}$$

Set

$$\gamma_1(W) = \frac{c_2 b_2 \gamma_0}{x_1(W)}.$$

$$a_1(W) = \gamma_1 - \gamma_0 + a_1 + 1 - c_2 \tag{16.41}$$
$$x_2(W) = \gamma_1(a_2 - c_3 - a_1(W)) + c_2(\gamma_0 + b_1 - a_2 + a_1(W)) \tag{16.42}$$
$$a_2(W) = -\gamma_1 + a_2 + c_2 - c_2. \tag{16.43}$$

The matrix has eigenvalues, $\theta, \theta_2, \theta_3$.

There are two feasibility conditions; γ_0, γ_1 are algebraic integers.

For arbitrary D, there are $D - 1$ feasibility conditions; $\gamma_0, \gamma_1, \dots, \gamma_{D-1}$ are algebraic integers.

Lemma 16.2. *With the notation of Theorem 16.1, suppose*

$$f_W = \frac{k - \lambda}{k} \quad (so,\ \ a_0(W) = -1).$$

Then,

$$a_i(W) = a_i + c_i - c_{i+1} \quad (0 \le i \le D - 1) \tag{16.44}$$
$$x_i(W) = b_i c_i \quad (1 \le i \le D - 1) \tag{16.45}$$
$$\gamma_i(W) = 0 \quad (0 \le i \le D - 1). \tag{16.46}$$

Proof. Since $\gamma_0 = a_0(W) + 1$, $\gamma_i = 0$. □

Chapter 17

Association Schemes

Monday, March 1, 1993

Review

Let $\Gamma = (X, E)$ be a distance-regular graph of diameter $D \geq 2$. Pick a vertex $x \in X$.

Let W be a thin irreducible $T(x)$-module with endpoint $r = 1$, diameter $d = D-1$ or $D-2$, and $r_0 = a_0(W) + 1$.

Show

$$\gamma_i = \frac{c_2 c_2 \cdots c_{i+1} b_2 b_3 \cdots b_{i+1} \gamma_0}{x_1(W) \cdots x_i(W)},$$

$a_i(W)$ and $x_i(W)$ are all algebraic integers in $\mathbb{Q}[\gamma_0]$, where

$$x_i(W) = c_i b_i + \gamma_{i-1}(a_i + c_i - c_{i+1} - a_{i-1}(W)) \qquad (1 \leq i \leq d) \qquad (17.1)$$

$$a_i(W) = \gamma_i - \gamma_{i-1} + a_i + c_i - c_{i+1} \qquad (1 \leq i \leq d) \qquad (17.2)$$

Certainly, $x_i(W)$, γ_i, and $a_i(W)$ are in $\mathbb{Q}[\gamma_0]$ by the above lines and so on.

$$\gamma_0 \to a_0(W) \to x_1(W) \to \gamma_1 \to a_1(W) \to x_1(W) \to \cdots.$$

Recall some $B \in \mathrm{Mat}_n(\mathbb{C})$ is integral whenever

$$B \in \mathrm{Mat}_n(\mathbb{Z}).$$

In this case, the characteristic polynomial

$$\det(\lambda I - B) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_0, \quad \text{for some } \ \alpha_0, \ldots, \alpha_{n-1} \in \mathbb{Z}.$$

Hence, eigenvalues of B are algebraic integers. But $a_i(W)$ is an eigenvalue of an integral matrices,

$$B = E^*_{i+1}(x) A E^*_{i+1}(x).$$

Hence, $a_i(W)$ is an algebraic integer.

Also, $x_i(W)$ is an eigenvalue of an integral matrix

$$B = E_i^*(x)AE_{i+1}^*(x)AE_i^*(x).$$

So $x_i(W)$ is an algebraic integer.

$$\gamma_i - \gamma_{i-1} = a_i(W) - a_i - c_i + c_{i+1}$$

is an algebraic integer.

Since $\gamma_0 = a_0(W) + 1$ is an algebraic integer, we find γ_i is an algebraic integer for all i.

Definition 17.1. A (commutative) association scheme is a configuration $Y = (X, \{R_i\}_{0\le i\le D})$, where X is a finite nonempty set (of vertices), $R_0, R_1, \dots, R_D$ are nonempty subsets of $X \times X$ such that

(*i*) $R_0 = \{(x,x) \mid x \in X\}$,

(*ii*) $R_0 \cup \cdots \cup R_D = X \times X$ (disjoint union),

(*iii*) for every i, $R_i^\top = \{(y,x) \mid xy \in R\} = R_{i'}$ for some $i' \in \{0, 1, \dots, D\}$,

(*iv*) for every h, i, j $(0 \le h, i, j \le D)$, and every $x, y \in X$ such that $(x,y) \in R_h$,

$$p_{ij}^h = |\{z \in X \mid (x,z) \in R_i,\ (z,y) \in R_j\}|$$

depends only on h, i, j and not on x, y; and

(*v*) $p_{ij}^h = p_{ji}^h$ for all h, i, j.

If $i' = i$ for all i, we say Y is symmetric. We call D the class of scheme and R_i, the ith relation of Y. We say vertices $x, y \in X$ are i-related, or ‘at distance i’, whenever $(x,y) \in R_i$.

We always assume that a ‘scheme’ is a commutative association scheme.

Let $Y = (X, \{R_i\}_{0\le i\le D})$ be an association scheme.

Definition 17.2. The i-the association matrix $A_i \in \mathbb{Mat}_X(\mathbb{C})$

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } (x,y) \in R_i \\ 0 & \text{if } (x,y) \notin R_i, \end{cases} \qquad (x, y \in X, 0 \le i \le D) \tag{17.3}$$

Then,

(*i′*) $A_0 = I$.

(*ii′*) $A_0 + A_1 + \cdots + A_D = J$ (= all 1’s matrix).

(*iii′*) $A_i^\top = A_{i'}$ $(0 \le i \le D)$.

(iv') $A_iA_j = \sum_{h=0}^{D} p_{ij}^h A_h \quad (0 \le i, j \le D).$

(v') $A_iA_j = A_jA_i$.

$M := \mathrm{Span}_{\mathbb{C}}(A_0, \dots, A_D)$ (Bose-Mesner algebra of Y) is a commutative $\mathbb{C}$-algebra of dimension $D+1$.

Observe:

$$Y \text{ is symmetric} \leftrightarrow A_i^\top = A_i \text{ for all } i \leftrightarrow M \text{ is symmetric.}$$

Example 17.1. Let $\Gamma = (X, E)$ be distance-regular of diameter D. Set

$$R_i = \{(x, y) \mid \partial(x, y) = i\} \qquad (0 \le i \le D). \tag{17.4}$$

Then,

$$Y = (X, \{R_i\}_{0 \le i \le D})$$

is a symmetric scheme.

$$i\text{-th association matrix} = i\text{-th distance matrix} \quad \text{for all } i.$$

Example 17.2. Suppose a group G acts transitively on a seet X. Assume G is generously transitive, i.e.,

$$\text{for all } x, y \in X, \text{ there exists } g \in G \text{ such that } gx = y, gy = x.$$

Then G acts on $X \times X$ by rule;

$$g(x, y) = (gx, gy), \quad \text{for all } \ g \in G, \text{ and for all } x, y \in X.$$

Let $R_0, \dots, R_D$ denote orbits of G on $X \times X$.

Observe that $R_i^\top = R_i$ for all i by generously transitivity, and

$$Y = (X, \{R_i\}_{0 \le i \le D})$$

is a symmetric scheme.

Exercise 17.1. In Example Example 17.2, Bose-Mesner algebra

$$M = \{B \in \mathrm{Mat}_X(\mathbb{C}) \mid Bg = gB, \text{ for all } g \in G\} \tag{17.5}$$
$$= \text{the commuting algebra of } G \text{ on } X. \tag{17.6}$$

Here, we view each $g \in G$ as a permutation matrix in $\mathrm{Mat}_X(\mathbb{C})$ satisfying

$$g\hat{x} = \widehat{gx} \quad \text{for all } x \in G.$$

Example 17.3. Let G be any finite group. G acts on $X = G$ by conjugation.

$$G \times X \to X, \quad (g, x) \mapsto gxg^{-1}.$$

Let $C_0, C_1, \dots, C_D$ denote orbits (i.e., conjugacy classes), and let $C_0 = \{1_G\}$. Claim that $Y = (X, \{R_i\}_{0 \le i \le D})$ is a commutative scheme (not symmetric in general).

(i) $R_0 = \{xx \mid x \in X\}$ as $C_0 = \{1_G\}$.

(ii) $R_0, \dots, R_D$ is a partition of $X \times X$ since $C_0, \dots, C_D$ is a partition of $X = G$.

(iii) $R_i^\top = R_{i'}$, where $C_{i'} = \{g^{-1} \mid g \in C_i\}$.

(iv) Set $H = G \oplus G$, the direct sum. Then H acts on $X = G$:

$$\text{for all } h = (g, gz), \text{for all } x \in X, \quad h(x) = gx(gx)^{-1} = gxz^{-1}g^{-1}.$$
$$R_i = \{(x, y) \mid x^{-1}y \in C_i\},\ h_i \in C_i,\ x^{-1}y = gh_ig^{-1}.$$

$$\begin{aligned}(x, y) &= (x, xgh_ig^{-1}) &(17.7)\\ &= (xgg^{-1}, xgh_ig^{-1}) &(17.8)\\ &= (xg, g)(1, h_i). &(17.9)\end{aligned}$$

So, $R_0, \dots, R_D$ are the orbits of H on $X \times X$.

(v) $p_{ij}^h = p_{ji}^h$?

Fix i, j, h and $x, y \in X$ with $(x, y) \in R_h$. Set

$$\begin{aligned}S &= \{z \in X \mid (x, z) \in R_i,\ (z, y) \in R_j\} &(17.10)\\ T &= \{z \in X \mid (x, z) \in R_j,\ (z, y) \in R_i\}. &(17.11)\end{aligned}$$

Show $|S| = |T|$.

$$\text{For all } z \in S,\ \text{set } \hat{z} = xz^{-1}y.$$

Observe, $\hat{z} \in T$.

$$\begin{aligned}&x^{-1}z \in C_i \quad x^{-1}\hat{z} = x^{-1}xz^{-1}y \in C_j &(17.12)\\ &z^{-1}y \in C_j \quad \hat{z}^{-1}y = y^{-1}zx^{-1}x^{-1}y = y^{-1}x(x^{-1}z)x^{-1}y \in C_i. &(17.13)\end{aligned}$$

Observe

$$S \to T \quad (z \mapsto z^{-1}) \quad \text{is one-to-one and onto.}$$

Chapter 18

Polynomial Schemes

Wednesday, March 3, 1993

Lemma 18.1. *Let* $Y = (X, \{R_i\}_{0 \le i \le D})$ *denote the symmetric scheme with associated matrices* $A_0, A_1, \ldots, A_D$*. Then the following are equivalent.*

(i) The graph $\Gamma = (X, R_1)$ *is distance-regular, and* $R_0, \ldots, R_D$ *are labelled so that*

$$R_i = \{xy \mid \partial(x, y) = i\}.$$

(ii) There exists $f_i \in \mathbb{C}[\lambda]$, $\deg f_i = i$ *such that* $f_i(A_1) = A_i$ *for all* i *with* $0 \le i \le D$.

(iii) The parameter p_{ij}^h

$$\begin{cases} = 0 & \textit{if one of } h, i, j \textit{ is larger than the sum of the other two,} \\ \ne 0 & \textit{if one of } h, i, j \textit{ is equal to the sum of the other two.} \end{cases}$$

Proof.

$(i) \Rightarrow (ii)$: Lemma 16.1.

$(ii) \Rightarrow (iii)$: Define

$$k_i \equiv p_{ii}^0 = |\{z \mid z \in X,\ \partial(x, z) = i,\ ((x, z) \in R_i)\}|$$

for any $x \in X$. Then $k_i \ne 0$ $(0 \le i \le D)$, $k_0 = 1$.

(By symmetricity, $(x, y) \in R_i$ if and only if $(y, x) \in R_i$.)

Claim.

$$
\begin{aligned}
k_h p^h_{ij} &= k_i p^i_{hj} = k_j p^j_{ih} && (18.1)\\
&= |X|^{-1}|\{xyz \in X^3 \mid \partial(x,y) = h, \partial(x,z) = i, \partial(y,z) = j\}|. && (18.2)
\end{aligned}
$$

Pf. The number of $xyz \in X^3$, $\partial(x,y) = h, \partial(x,z) = i, \partial(y,z) = j$ is equal to

$$|X|k_h p^h_{ij} = |X|k_i p^i_{hj} = k_j p^j_{ih}.$$

In particular,

$$p^h_{ij} = 0 \leftrightarrow p^i_{hj} = 0 \leftrightarrow p^j_{ih} = 0.$$

Hence, it suffices to show

$$\begin{cases} p^h_{ij} = 0 & \text{if } \ h > i + j \\ p^h_{ij} \neq 0 & \text{if } \ h = i + j. \end{cases}$$

Fix i, j. Without loss of generality, we may assume that $i + j \leq D$ as trivial otherwise.

$$f_i(A)f_j(A) = A_i A_j = \sum_{\ell=0}^{D} p^\ell_{ij} A_\ell = \sum_{\ell=0}^{D} p^\ell_{ij} f_\ell(A).$$

$$
\begin{aligned}
i + j &= \deg \mathrm{LHS} && (18.3)\\
&= \deg \mathrm{RHS} && (18.4)\\
&= \max\{\ell \mid p^\ell_{ij} \neq 0\}. && (18.5)
\end{aligned}
$$

(iii) $\Rightarrow$ *(i)*

Let $A = A_1$, and consider a graph Γ with adjacency matrix A.

$$
\begin{aligned}
AA_j &= \sum_h p^h_{1j} A_h && (18.6)\\
&= p^{j+1}_{1j} A_{j+1} + p^j_{1j} A_j + p^{j-1}_{1j} A_{j-1}. && (18.7)
\end{aligned}
$$

Then, $p^{j+1}_{1j} \neq 0 \neq p^{j-1}_{1j}$.

Fix a vertex $x \in X$, and set $R_i(x) = \{y \mid (x,y) \in R_i\}$.

Then each $y \in R_i(x)$ is adjacent in Γ to exactly

$$
\begin{aligned}
&p^i_{1,i+1} \neq 0 \quad \text{vertices in } \ R_{i+1}(x), && (18.8)\\
&p^i_{1i} \qquad\ \ \text{vertices in } \ R_i(x), && (18.9)\\
&p^i_{1,i-1} \neq 0 \quad \text{vertices in } \ R_{i-1}(x). && (18.10)
\end{aligned}
$$

Hence, by induction,

$$R_i(x) = \{y \mid \partial(x,y) = i \text{ in } \Gamma\} \qquad (0 \leq i \leq D), \qquad (18.11)$$

and Γ is distance regular.

□

Chapter 19

Commutative Association Schemes

Friday, March 5, 1993

Lemma 19.1. *Let $Y = (X, \{R_i\}_{0 \le i \le D})$ be a commutative scheme with Bose-Mesner algebra M.*

Then there exists a basis $E_0, E_1, \dots, E_D$ for M such that

(i) $E_0 = |X|^{-1}J$.

(ii) $E_iE_j = E_jE_i = \delta_{ij}E_i \quad (0 \le i, j \le D)$.

(iii) $E_0 + E_1 + \cdots + E_D = I$.

(iv) $E_i^\top = \overline{E_i} = E_{\hat{i}}$ *for some* $\hat{i} \in \{0, 1, \dots, D\}$.

Proof. M acts on Hermitean space $V = \mathbb{C}^n$ $(n = |X|)$.

If W is an M-module, so is $W^\perp$.

Each irreducible M-module is 1 dimensional by commutativity of M. So V is orthognal direct sum of 1-dimensional M-modules.

Let $v_1, \dots, v_n$ be an orthonormal basis for V consisiting of eigenvectors for all $m \in M$.

Set $P \in \mathrm{Mat}_X(\mathbb{C})$ so that the i-th column of P is equal to v_i. So,

$$\bar{P}^\top P = I = P\bar{P}^\top = \bar{P}P^\top,$$

and P is unitary.

Also, for all $m \in M$,

$$\begin{aligned} P^{-1}mP &= \text{diagonal} && (19.1)\\ &= \operatorname{diag}(\theta_1(m), \dots, \theta_n(m)). && (19.2) \end{aligned}$$

for some functions

$$\theta_i : M \longrightarrow \mathbb{C}.$$

Observe: each $\theta = \theta_i$ is a character of M, i.e.,

$$\theta : M \longrightarrow \mathbb{C}$$

is a $\mathbb{C}$-algebra homomorphism.

Observe: the $\theta_1, \dots, \theta_n$ are not all distinct.

Let $\sigma_0, \dots, \sigma_r$ denote distinct elements of

$$\theta_1, \dots, \theta_n.$$

Say σ_i appears m_i times. Without loss of generality, we may assume that

$$P^{-1}mP = \begin{pmatrix} \sigma_0(m)I_{m_0} & O & O & O \\ O & \sigma_1(m)I_{m_1} & O & O \\ O & O & \ddots & O \\ O & O & O & \sigma_r(m)I_{m_r} \end{pmatrix}.$$

Set

$$E_i = P \begin{pmatrix} O & O & O \\ O & I_{m_i} & O \\ O & O & O \end{pmatrix} P^{-1},$$

where I_{m_i} is in the i-th block.

Then,

$$E_iE_j = \delta_{ij}E_i \quad (0 \le i, j \le r),$$
$$E_0 + E_1 + \cdots + E_r = I.$$

Hence for all $m \in M$,

$$m = \sum_{i=0}^{r} \sigma_i(m)E_i \in \operatorname{Span}(E_0, \dots, E_r).$$

So,

$$M \subseteq \operatorname{Span}(E_0, \dots, E_r).$$

Since $E_0, \dots, E_r$ are linearly independent, $r \ge D$.

Show $E_i \in M$.

Claim 1. For all distinct $i, j \quad (0 \le i, j \le D)$, there exists $m \in M$ such that $\sigma_i(m) \ne 0$, $\sigma_j(m) = 0$.

Pf of Claim 1. $\sigma_i \neq \sigma_j$ implies that there exists $m' \in M$ such that $\sigma_i(m') \neq \sigma_j(m')$.

Set $m = m' - \sigma_j(m')I$. Then,

$$\sigma_j(m) = \sigma_j(m') - \sigma_j(m') = 0, \tag{19.3}$$
$$\sigma_i(m) = \sigma_i(m') - \sigma_j(m') \neq 0. \tag{19.4}$$

Claim 2. $E_i \in M \quad (0 \leq i \leq D)$.

Pf of Claim 2. Fix a vertex $x \in X$. For all $j \neq i$, there exists $m_j \in M$ such that

$$\sigma_i(m_j) \neq 0, \quad \sigma_j(m_j) = 0, \quad i \neq j.$$

Observe

$$s = \sigma_i\left(\prod_{\ell \neq i} m_\ell\right) \neq 0.$$

Set

$$m^* = \left(\prod_{\ell \neq i} m_\ell\right) s^{-1}.$$

Observe

$$\sigma_i(m^*) = 1, \quad \sigma_j(m^*) = 0, \quad \text{for all } j \neq i \quad (0 \leq j \leq D).$$

So

$$P^{-1}m^*P = \begin{pmatrix} O & O & O \\ O & I_{m_i} & O \\ O & O & O \end{pmatrix}.$$

We have

$$E_i = m^* \in M.$$

Now $r = D$, $M = \operatorname{Span}(E_0, \dots, E_D)$ and $E_0, \dots, E_D$ is a basis for M.

Observe

$$P^{-1}E_iP = \begin{pmatrix} O & O & O \\ O & I_{m_i} & O \\ O & O & O \end{pmatrix}$$

implies

$$P^{-1}\overline{E_i}^\top P = \bar{P}^\top \overline{E_i}^\top \overline{P^{-1}}^\top = \begin{pmatrix} O & O & O \\ O & I_{m_i} & O \\ O & O & O \end{pmatrix}^\top = P^{-1}E_iP.$$

Hence,

$$\overline{E_i}^\top = E_i.$$

$E_0^\top, \dots, E_D^\top$ are nonzero matrices satisfying

$$E_i^\top E_j^\top = \delta_{ij} E_i^\top,$$

$$E_0^\top + E_1^\top + \cdots + E_D^\top = I.$$

Each $E_i^\top$ is a linear combination of $E_0, \dots, E_D$ with coefficientss that are 0 or 1, and for no two E_i's are coefficients of any E_j both 1's.

So, $E_0^\top, \dots, E_D^\top$ is a permutation of $E_0, \dots, E_D$.

Observe $J = A_0 + \cdots + A_D \in M$.

The matrix $|X|^{-1}J$ is an idempotent of rank 1.

So, without loss of generality we may assume that

$$E_0 = \frac{1}{|X|} J.$$

We have the assertions. □

Define entry-wise product $\circ$ on $\mathrm{Mat}_X(\mathbb{C})$.

$$A_i \circ A_j = \delta_{ij} A_i.$$

So, M is closed under $\circ$.

$$E_i \circ E_j = \frac{1}{|X|} \sum_{h=0}^{D} q_{ij}^h E_h.$$

The numbers q_{ij}^h is called Krein parameters of Y.

Claim. $q_{ij}^h \in \mathbb{R}$.

Pf.

$$\begin{aligned}
\frac{1}{|X|} \sum_{h=0}^{D} \overline{q_{ij}^h} E_h &= \frac{1}{|X|} \sum_{h=0}^{D} \overline{q_{ij}^h} \overline{E_h}^\top && (19.5)\\
&= (\overline{E_i \circ E_j})^\top && (19.6)\\
&= E_i \circ E_j && (19.7)\\
&= \frac{1}{|X|} \sum_{h=0}^{D} q_{ij}^h E_h. && (19.8)
\end{aligned}$$

Hence, $q_{ij}^h = \overline{q_{ij}^h}$.

Observe $A_0, \dots, A_D$, $E_0, \dots, E_D$ are bases for M. Hence, there exist $p_i(j)$, $q_i(j) \in \mathbb{C}$ such that

$$A_i = \sum_{j=0}^{D} p_i(j) E_j \tag{19.9}$$

$$E_i = \frac{1}{|X|} \sum_{j=0}^{D} q_i(j) A_j. \tag{19.10}$$

Taking transpose and conjugate we find,

$$\overline{p_i(j)} = p_i(j) = p_{i'}(\hat{j}) \qquad (0 \le i, j \le D) \tag{19.11}$$

$$\overline{q_i(j)} = q_i(j) = q_{\hat{i}}(j') \qquad (0 \le i, j \le D). \tag{19.12}$$

Fix a vertex $x \in X$. Define

$$E_i^* \equiv E_i^*(x) \in \mathrm{Mat}_X(\mathbb{C})$$

to be a diagonal matrix such that

$$(E_i^*)_{xy} = \begin{cases} 1 & \text{if } (x,y) \in R_i \\ 0 & \text{if } (x,y) \notin R_i \end{cases} \quad (0 \le i \le D, y \in X.)$$

Then,

$$E_i^* E_j^* = \delta_{ij} E_i^*,$$
$$E_0^* + \dots + E_D^* = I,$$
$$(E_i^*)^\top = \overline{E_i^*} = E_i^*.$$

Definition 19.1. Dual Bose-Mesner algebra : $M^* \equiv M^*(x)$ with respect to x is

$$\mathrm{Span}(E_0^*, \dots, E_D^*).$$

Define dual associate matrices $A_0^*, \dots, A_D^*$. Indeed $A_i^* \equiv A_i^*(x) \in \mathrm{Mat}_X(\mathbb{C})$ is a diagonal matrix with

$$(A_i^*)_{yy} = |X| (E_i)_{xy} \quad (y \in X).$$

A_i^* is a diagonal matrix having the row x of E_i^* on the diagonal.

Observe

$$A_i^* = \sum_{j=0}^{D} q_i(j) E_j^* \quad \left(E_i = \frac{1}{|X|} \sum_{j=0}^{D} q_i(j) A_j \right) \tag{19.13}$$

$$E_i^* = \frac{1}{|X|} \sum_{j=0}^{D} p_i(j) A_j^* \quad \left(A_i = \sum_{j=0}^{D} p_i(j) E_j \right). \tag{19.14}$$

So, $A^*_0, \dots, A^*_D$ form a basis for M^*.

Also,

$$A^*_i E^*_j = q_i(j) E^*_j.$$

$$\left(A^*_i E^*_j = \sum_{h=0}^{D} q_i(h) E^*_h E^*_j = q_i(j) E^*_j. \right)$$

So, $q_i(j)$ are dual eigenvalues of A^*_i.

Observe,

$$A^*_0 = I, \quad A^*_0 + \dots + A^*_D = |X| E^*_0, \quad \overline{A^*_i} = A^*_{\hat{i}},$$

$$A^*_i A^*_j = \sum_{h=0}^{D} q^h_{ij} A^*_h \quad (0 \le i, j \le D).$$

HS MEMO

Proof.

$$(A^*_0)_{yy} = |X| (E_0)_{xy} = (J)_{xy} = 1.$$

$$A^*_0 + \dots + A^*_D = \sum_{i=0}^{D} \sum_{j=0}^{D} q_i(j) E^*_j = |X| E^*_0.$$

Note that

$$I = E_0 + \dots + E_D = \frac{1}{|X|} \sum_{i=0}^{D} \sum_{j=0}^{D} q_i(j) A_j.$$

$$\sum_{i=0}^{D} q_i(j) = \delta_{j0} |X|.$$

$$\overline{A^*_i} = \sum_{j=0}^{D} \overline{q_i(j)} E^*_j = \sum_{j=0}^{D} q_{\hat{i}}(j) E^*_j = A^*_{\hat{i}}.$$

$$(A^*_i A^*_j)_{yy} = |X|^2 (E_i)_{xy} (E_j)_{xy} \tag{19.15}$$

$$= |X|^2 (E_i \circ E_j)_{xy} \tag{19.16}$$

$$= |X| \sum_{h=0}^{D} q^h_{ij} (E_h)_{xy} \tag{19.17}$$

$$= \sum_{h=0}^{D} q^h_{ij} (A^*_h)_{yy}. \tag{19.18}$$

The following statements will be proved after a couple of lemmas in the next lecture.

Lemma. Let $Y = (X, \{R_i\}_{0 \le i \le D})$ be a commutative scheme. Fix a vertex $x \in X$, and set $E^* \equiv E^*_i(x)$ and $A^*_i \equiv A^*(x)$. Then the following hold.

(*i*) $E_i^* A_j E_k^* = O$ if and only if $p_{ij}^k = 0$ for $0 \leq i, j, k \leq D$.

(*ii*) $E_i A_j^* E_k = O$ if and only if $q_{ij}^k = 0$ for $0 \leq i, j, k \leq D$.

Chapter 20

Vanishing Conditions

Monday, March 15, 1993 (Monday after Spring break)

Lemma 20.1. *Let $Y = (X, \{R_i\}_{0 \le i \le D})$ be a commutative scheme.*

(i) $p_0(i) = 1$.

(ii) $p_i(0) = k_i$, *where*

$$k_i = p^0_{ii'} = |\{y \in X \mid (x, y) \in R_i\}| \quad (x \in X).$$

(iii) $q_0(i) = 1$.

(iv) $q_i(0) = m_i$, *where*

$$m_i = \mathrm{rank} E_i.$$

Proof.

(i) Since $A_0 = I$ and

$$A_0 = p_0(0)E_0 + p_0(1)E_1 + \cdots + p_0(D)E_D \tag{20.1}$$
$$I = E_0 + E_1 + \cdots + E_D, \tag{20.2}$$

$p_0(i) = 1$ for all i.

(ii) Since

$$A_i = p_i(0)E_0 + p_i(1)E_1 + \cdots + p_i(D)E_D,$$

$A_i E_0 = p_i(0)E_0$, and

$$k_i J = A_i J = p_i(0)J$$

as there are k_i 1's in each row of A_i, we have $k_i = p_i(0)$.

(iii) Since $E_0 = |X|^{-1}J$ and

$$E_0 = |X|^{-1}(q_0(0)A_0 + q_0(1)A_1 + \cdots + q_0(D)A_D) \tag{20.3}$$

$$|X|^{-1}J = |X|^{-1}(A_0 + A_1 + \cdots + A_D), \tag{20.4}$$

$q_0(i) = 1$ for all i.

(iv) $E_i = |X|^{-1}(q_i(0)A_0 + q_i(1)A_1 + \cdots + q_i(D)A_D)$, $E_i^2 = E_i$, and E_i is similar to a matrix

$$\begin{pmatrix} I_{m_i} & O \\ O & O \end{pmatrix}.$$

So,

$$m_i = \mathrm{rank}E_i = \mathrm{trace}E_i = \sum_{x \in X}(E_i)_{xx} = |X||X|^{-1}q_i(0) = q_i(0).$$

Note that as

$$E_i = \frac{1}{|X|}\sum_{j=0}^{D} q_i(j)A_j \to (E_i)_{xx} = \frac{1}{|X|}q_i(0)(A_0)_{xx}.$$

Hence, we have all formulas.

□

Lemma 20.2. *With the above notation*

(i) $p_{ij}^h = p_{j'i'}^{h'}$.

(ii) $k_h p_{ij}^h = k_j p_{i'h}^j = k_i p_{hj'}^i$.

(iii) $q_{ij}^h = q_{\hat{j}\hat{i}}^{\hat{h}}$.

(iv) $m_h q_{ij}^h = m_j q_{\hat{i}h}^j = m_i q_{h\hat{j}}^i$.

Proof.

(i) We have

$$\sum_{h=0}^{D} p_{ij}^{h} A_{h'} = \left(\sum_{h=0}^{D} p_{ij}^{h} A_{h} \right)^{\top} \tag{20.5}$$

$$= (A_i A_j)^{\top} \tag{20.6}$$

$$= A_j^{\top} A_i^{\top} \tag{20.7}$$

$$= A_{j'} A_{i'} \tag{20.8}$$

$$= \sum_{h=0}^{D} p_{j'i'}^{h'} A'_{h}. \tag{20.9}$$

(*ii*) Count the following number,

$$|\{xyz \in X^3 \mid (x,y) \in R_h, (x,z) \in R_i, (z,y) \in R_j\}| \tag{20.10}$$

$$= |X| k_h p_{ij}^{h} = |X| k_j p_{i'h}^{j} = |X| k_{hj'}^{i}. \tag{20.11}$$

(*iii*)

$$\frac{1}{|X|} \sum_{h=0}^{D} q_{ij}^{h} E_{\hat{h}} = \left(\frac{1}{|X|} \sum_{h=0}^{D} q_{ij}^{h} E_{h} \right)^{\top} \tag{20.12}$$

$$= (E_i \circ E_j)^{\top} \tag{20.13}$$

$$= E_j^{\top} \circ E_i^{\top} \tag{20.14}$$

$$= E_{\hat{j}} E_{\hat{i}} \tag{20.15}$$

$$= \frac{1}{|X|} \sum_{h=0}^{D} q_{\hat{j}\hat{i}}^{\hat{h}} E_{\hat{h}}. \tag{20.16}$$

(*iv*) Let $\tau(B)$ denote the sum of the entries in the matrix B.

Observe: $\tau(B \circ C) = \mathrm{trace}(BC^{\top})$.

Observe

$$\tau(E_i \circ E_j \circ E_{\hat{k}}) = \tau((E_i \circ E_j \circ E_{\hat{k}})^{\top}) = \tau(E_{\hat{i}} \circ E_k \circ E_{\hat{j}}) = \tau(E_k \circ E_{\hat{j}} \circ E_{\hat{i}}).$$

Compute each one.

$$\tau(E_i \circ E_j \circ E_{\hat{k}}) = \operatorname{trace}((E_i \circ E_j)E_k) = \operatorname{trace}\left(\left(\frac{1}{|X|}\sum_h q_{ij}^h E_h\right) E_k\right) \tag{20.17}$$

$$= \operatorname{trace}\left(\frac{1}{|X|} q_{ij}^k E_k\right) = \frac{1}{|X|} m_k q_{ij}^k, \tag{20.18}$$

$$\tau(E_{\hat{i}} \circ E_k \circ E_{\hat{j}}) = \operatorname{trace}((E_{\hat{i}} \circ E_k)E_{\hat{j}}) = \operatorname{trace}\left(\left(\frac{1}{|X|}\sum_h q_{\hat{i}k}^h E_h\right) E_{\hat{j}}\right) \tag{20.19}$$

$$= \operatorname{trace}\left(\frac{1}{|X|} q_{\hat{i}k}^j E_k\right) = \frac{1}{|X|} m_j q_{\hat{i}k}^j, \tag{20.20}$$

$$\tau(E_k \circ E_{\hat{j}} \circ E_{\hat{i}}) = \operatorname{trace}((E_k \circ E_{\hat{j}})E_i) = \operatorname{trace}\left(\left(\frac{1}{|X|}\sum_h q_{k\hat{j}}^h E_h\right) E_i\right) \tag{20.21}$$

$$= \operatorname{trace}\left(\frac{1}{|X|} q_{k\hat{j}}^i E_i\right) = \frac{1}{|X|} m_i q_{k\hat{j}}^i. \tag{20.22}$$

Hence, we have (iv).

□

Lemma 20.3. *Let $Y = (X, \{R_i\}_{0\le i\le D})$ be a commutative scheme. Fix a vertex $x \in X$, and set $E_i^* \equiv E_i^*(x)$ and $A_i^* \equiv A^*(x)$. Then the following hold.*

(i) $E_i^* A_j E_k^* = O$ if and only if $p_{ij}^k = 0$ for $0 \le i, j, k \le D$.

(ii) $E_i A_j^* E_k = O$ if and only if $q_{ij}^k = 0$ for $0 \le i, j, k \le D$.

Proof.

(i) Partition rows and columns by $R_0(x), R_1(x), \dots, R_D(x)$. Then,

$$E_i^*(x) A_j E_h^*(x)$$

is the (i, h) block of A_j.

Hence this submatrix is zero if and only if there exists no $y, z \in X$ such that $(x, y) \in R_i$, $(x, z) \in R_h$ and $(y, z) \in R_j$. This is exactly when $p_{ij}^h = 0$.

(ii) The sum of the squares of norms of entries in $E_i A_j^* E_k$

$$= \tau((E_iA^*_jE_k) \circ (\overline{E_jA^*_jE_k})) \tag{20.23}$$
$$= \operatorname{trace}(E_iA^*_jE_k(\overline{E_jA^*_jE_k})^\top) \tag{20.24}$$
$$= \operatorname{trace}(E_iA^*_jE_kA^*_{\hat{j}}E_i) \tag{20.25}$$
$$= \operatorname{trace}(E_iA^*_jE_kA^*_{\hat{j}}) \qquad \text{as } \operatorname{trace}(XY) = \operatorname{trace}(YX) \tag{20.26}$$
$$= \sum_{y\in X}(E_iA^*_jE_kA^*_{\hat{j}})_{yy} \tag{20.27}$$
$$= \sum_{y\in X}\left(\sum_{z\in X}(E_i)_{yz}(A^*_j)_{zz}(E_k)_{zy}(A^*_{\hat{j}})_{yy}\right) \tag{20.28}$$
$$= \sum_{y\in X}\left(\sum_{z\in X}(E_{\hat{i}})_{zy}(|X|(E_j)_{xz})(E_k)_{zy}(|X|(E_j)_{yx})\right) \tag{20.29}$$
$$= |X|^2(E_j(E_{\hat{i}} \circ E_k))E_j)_{xx} \tag{20.30}$$
$$= |X|q^j_{\hat{i}k}(E_j)_{xx} \tag{20.31}$$
$$= q^j_{\hat{i}k}m_j \tag{20.32}$$
$$= m_kq^k_{ij}. \tag{20.33}$$

Note that since $|X|E_j = q_j(0)A_0 + q_j(1)A_1 + \cdots q_j(D)A_D$,

$$(E_j)_{xx} = \frac{1}{|X|}q_j(0) = \frac{m_j}{|X|}.$$

Thus, we have (ii).

□

Corollary 20.1 (Krein Condition)**.** *For any commutative scheme $Y = (X, \{R_i\}_{0\le i\le D})$, q^h_{ij} is a non-negative real number for $0 \le h, i, j \le D$.*

Proof. Since $q^h_{ij}m_h$ is a non-negative real by the proof of Lemma 20.3 (ii).

Note that m_h is a positive integer. □

An interpretation of the Krein parameters.

Let $Y = (X, \{R_i\}_{0\le i\le D})$ be a commutative scheme with standard module V.

Pick a vector $v \in V$ with

$$v = \sum_{x\in X} \alpha_x \hat{x}.$$

View v as a function

$$v : X \longrightarrow \mathbb{C} \quad (x \mapsto \alpha_x).$$

View V as the set of all functions $V \longrightarrow \mathbb{C}$. Then the vector space V together with product of functions is a $\mathbb{C}$-algebra.

For

$$v = \sum_{x \in X} \alpha_x \hat{x}, \quad w = \sum_{x \in X} \beta_x \hat{x} \in V,$$

write

$$v \circ w = \sum_{x \in X} \alpha_x \beta_x \hat{x}$$

to represent the product of v and w viewed as functions.

Lemma 20.4. *With the above notation,*

(i) $A^*_j(x)v = |X|(E_{\hat{j}}\hat{x} \circ v)$ *for all* $v \in V$ *and for all* $x \in X$.

(ii) $E_i V \circ E_j V \subseteq \sum_{h: q^h_{ij} \neq 0} E_h V$ *for all* $0 \leq i, j \leq D$.

(iii) $E_h(E_i \circ E_j V) = E_h V$ *if* $q^h_{ij} \neq 0$ *for all* $0 \leq h, i, j \leq D$.

Chapter 21

Norton Algebras

Wednesday, March 17, 1993

Proof of Lemma 20.4.

(i) Suppose

$$v = \sum_{x \in X} \alpha_x \hat{x}.$$

Pick a vertex $z \in X$ and compare z-coordinate of each side in (i).

$$(A_j^*(x)v)_z = (A_j^*(x))_{zz} v_z = |X|(E_j)_{xz} \alpha_z. \tag{21.1}$$
$$|X|(E_{\hat{j}} \hat{x} \circ v)_z = |X|(E_{\hat{j}} \hat{x})_z \cdot \alpha_z = |X|(E_j)_{xz} \alpha_z. \tag{21.2}$$

Note that $E_{\hat{j}} \hat{x}$ is the column x of $E_{\hat{j}}$, which is the row x of E_j.

(ii) Fix i, j, h such that $q_{ij}^h = 0$.

Claim. $E_h(E_i V \circ E_j V) = 0$.

$$\begin{aligned}
E_h(E_i V \circ E_j V) &= E_h(\mathrm{Span}(v \circ w \mid v \in E_i V, w \in E_j V)) && (21.3)\\
&= E_h(\mathrm{Span}(E_i \hat{y} \circ E_j \hat{z} \mid y, z \in X)) && (21.4)\\
&= \mathrm{Span}(E_h(E_j \hat{z} \circ E_i \hat{y} \mid y, z \in X) && (21.5)\\
&= \mathrm{Span}((E_h A_{\hat{j}}^*(z) E_i)\hat{y} \mid y, z \in X) \qquad \text{by } (i). && (21.6)
\end{aligned}$$

But $q_{ij}^h = 0$ implies $q_{\hat{j}\hat{i}}^{\hat{h}} = 0$.

So, by Lemma 20.3 (ii),

$$0 = (E_{\hat{i}} A_{\hat{j}}^* E_{\hat{h}})^\top = E_h A_{\hat{j}}^* E_i.$$

Hence, $E_h(E_iV \circ E_jV) = 0$.

(iii) Fix i, j, h such that $q^h_{ij} \neq 0$. Then,

$$E_h(E_iV \circ E_jV) \subseteq E_hV$$

is clear. We show the other inclusion. Since

$$\begin{aligned}
E_i\hat{y} \circ E_j\hat{y} &= (\text{column } y \text{ of } E_i \circ \text{column } y \text{ of } E_j) && (21.7)\\
&= \text{column } y \text{ of } E_i \circ E_j && (21.8)\\
&= (E_i \circ E_j)\hat{y} && (21.9)\\
&= \left(\frac{1}{|X|}\sum_{h=0}^{D} q^h_{ij}E_h\right)\hat{y}, && (21.10)
\end{aligned}$$

we have,

$$\begin{aligned}
E_h(E_iV \circ E_jV) &= E_h\text{Span}(E_i\hat{y} \circ E_j\hat{z} \mid y, z \in X) && (21.11)\\
&\supseteq E_h\text{Span}(E_i\hat{y} \circ E_j\hat{y} \mid y \in X) && (21.12)\\
&= \text{Span}(q^h_{ij}E_h\hat{y} \mid y \in X) && (21.13)\\
&= \text{Span}(E_h\hat{y} \mid y \in X) \qquad \text{since } q^h_{ij} \neq 0 && (21.14)\\
&= E_hV. && (21.15)
\end{aligned}$$

This proves the assertion.

□

Lemma 21.1. *Given a commutative scheme $Y = (X, \{R_i\}_{0\le i\le D})$, fix j $(0 \le j \le D)$. Define a binary multiplication:*

$$E_jV \times E_jV \longrightarrow E_jV \quad ((v, w) \mapsto v * w = E_j(v \circ w)).$$

Then,

(i) $v * w = w * v$, for all $v, w \in E_jV$,

(ii) $v * (w + w') = v * w + v * w'$ for all $v, w, w' \in E_jV$, and

(iii) $(\alpha v) * w = \alpha(v * w)$ for all $\alpha \in \mathbb{C}$.

In particular, the vector space E_jV together with $*$ is a commutative $\mathbb{C}$-algebra, (not associative in general).

($N_j : (E_jV, *)$ is called the Norton algebra on E_jV.)

(iv) $v * w = 0$ for all $v, w \in E_jV$ if and only if $q^j_{jj} = 0$.

Proof.

$(i) - (iii)$ Immediate.

(iv) Immediate from Lemma 20.4 (ii), (iii).

$\square$

Let Y, j, N_j be as in Lemma 21.1, and M Bose-Mesner algebra of Y. Let

$$\mathrm{Aut}Y = \{\sigma \in \mathrm{Mat}_X(\mathbb{C}) \mid \sigma : \text{ permutation matrix }, \sigma \cdot m = m \cdot \sigma \text{ for all } m \in M\} \tag{21.16}$$

$$= \{\sigma \in \mathrm{Mat}_X(\mathbb{C}) \mid \sigma : \text{ permutation matrix }, \tag{21.17}$$

$$(x,y) \in R_i \to (\sigma x, \sigma y) \in R_i, \text{ for all } i, \text{ and for all } x, y \in X\} \tag{21.18}$$

$$\mathrm{Aut}(N_j) = \{\sigma : E_jV \to E_jV \mid \sigma \text{ is a } \mathbb{C}\text{-algebra isomorphim}, i.e., \tag{21.19}$$

$$\sigma(v * w) = \sigma(v) * \sigma(w) \text{ for all } v, w \in E_jV\}. \tag{21.20}$$

Lemma 21.2. *Let $Y, j, *$ be as in Lemma 21.1.*

(i) E_jV *is a module for* $\mathrm{Aut}(Y)$.

(ii) $\sigma|_{E_jV} \in \mathrm{Aut}(N_j)$ *for all* $\sigma \in \mathrm{Aut}(Y)$.

(iii) $\mathrm{Aut}Y \longrightarrow \mathrm{Aut}(N_j)$, $(\sigma \mapsto \sigma|_{E_j})$ *is a homomorphism of groups,*

(i.e., a representation of $\mathrm{Aut}(Y)$*).*

(iv) *Suppose $R_0, \dots, R_D$ are orbits of* $\mathrm{Aut}(Y)$ *acting on $X \times X$, (so, we are in Example 17.2) then above representation is irreducible.*

Proof.

(i) Pick $\sigma \in \mathrm{Aut}Y$ and $v \in V$. Then,

$$\sigma E_j v = E_j \sigma v,$$

since σ commutes with each element of M.

(ii) $\sigma|_{E_jV} : E_jV \to E_jV$ is an isomorphism of a vector space. Since σ is invertible,for all $v, w \in E_jV$,

$$\sigma(v*w) = \sigma(E_j(E_jv \circ E_jw)) = E_j\sigma(E_jv \circ E_jw) = E_j(E_j\sigma v \circ E_j\sigma w) = \sigma(v)*\sigma(w).$$

(iii) Immediate from (i) and (ii).

(iv) Here, Bose-Mesner algebra M is the full commuting algebra, i.e.,

$$M = \{m \in \mathrm{Mat}_X(\mathbb{C}) \mid \sigma \cdot m = m \cdot \sigma, \text{ for all } \sigma \in \mathrm{Aut}(Y)\}.$$

Suppose there sia a nonzero proper subspace $0 \neq W \subsetneq E_j V$ that is $\mathrm{Aut}(Y)$-invariant.

Set

$$W^{\perp} = \{v \in E_j V \mid \langle w, v \rangle = 0, \text{ for all } w \in W\}.$$

Then, $W^{\perp}$ is a module for $\mathrm{Aut}(Y)$, since $\mathrm{Aut}(Y)$ is closed under transpose conjugate.

Let $e : V \to W$ and $f : V \to W^{\perp}$ be the orthogonal projection such that $e + f = E_j$,

$$e^2 = e, f^2 = f, ef = fe = 0, eE_h = 0, \text{ if } h \neq j.$$

Since e commutes with all $\sigma \in \mathrm{Aut}(Y)$, $e \in M$ and

$$e = \sum_{i=0}^{D} \alpha_i E_i.$$

If $h \neq j$, then $0 = eE_h$ and $\alpha_h = 0$. Thus, $e = \alpha_j E_j$, i.e., $e = 0$ or $f = 0$.

A contradiction.

$\square$

Norton algebras were used in original construction of Monster, a finite simple group G.

Compute character table of G,

$\to$ p_{ij}^h, q_{ij}^h of group scheme on G,

$\to$ find j where $m_j = \dim E_j V$ is small and $q_{jj}^j \neq 0$,

$\to$ guess abstract structure of N_j using the knowlege of p_{ij}^h's and q_{ij}^h's,

$\to$ compute $\mathrm{Aut}(N_j)$,

$\to$ G.

Chapter 22

Q-Polynomial Schemes

Friday, March 19, 1993

Lemma 22.1. *Let $Y = (X, \{R_i\}_{0\le i\le D})$ be a commutative scheme.*

(i) $p^h_{0j} = p^h_{j0} = \delta_{jh}..$

(ii) $p^0_{ij} = \delta_{ij'} k_i$.

(iii) $q^h_{0j} = q^h_{j0} = \delta_{jh}$.

(iv) $q^0_{ij} = \delta_{i\hat{j}} m_i$.

(v) $\displaystyle\sum_{j=0}^{D} p^h_{ij} = k_i$.

(vi) $\displaystyle\sum_{j=0}^{D} q^h_{ij} = m_i$.

Proof.

(i), (ii) These are trivial.

(iii) We have

$$|X|^{-1}\sum_{\ell=0}^{D} q^\ell_{0j}E_\ell = E_0 \circ E_j = |X|^{-1}J \circ E_j = |X|^{-1}E_j.$$

(iv) Recall from Lemma 20.2

$$|X|^{-1}m_h q^h_{ij} = \tau(E_i \circ E_j \circ E_{\hat{h}}),$$

(where $\tau(B)$ is the sum of entries in matrix B.)

$$\begin{aligned}
|X|^{-1}m_0 q_{ij}^0 &= \tau(E_i \circ E_j \circ E_0) && (22.1)\\
&= |X|^{-1}\tau(E_i \circ E_j) \qquad (E_0 = |X|^{-1}J) && (22.2)\\
&= |X|^{-1}\mathrm{trace}(E_i E_{\hat{j}}) && (22.3)\\
&= |X|^{-1}\delta_{i\hat{j}}\mathrm{trace}E_i && (22.4)\\
&= |X|^{-1}\delta_{i\hat{j}} m_i. && (22.5)
\end{aligned}$$

(v) Pick $x, y \in X$ with $(x, y) \in R_h$. Then,

$$\begin{aligned}
\sum_{j=0}^{D} p_{ij}^h &= |\{z \in X \mid (x,z) \in R_i,\ (z,y) \in R_j \text{ for some } j\}| && (22.6)\\
&= |\{z \in X \mid (x,z) \in R_i\}| && (22.7)\\
&= k_i. && (22.8)
\end{aligned}$$

(vi)

$$E_i \circ E_j = |X|^{-1}\sum_{h=0}^{D} q_{ij}^h E_h.$$

So,

$$\begin{aligned}
\sum_{j=0}^{D} E_i \circ E_j &= |X|^{-1}\sum_{h=0}^{D}\left(\sum_{j=0}^{D} q_{ij}^h\right) E_h && (22.9)\\
&= E_i \circ \sum_{j=0}^{D} E_j && (22.10)\\
&= E_i \circ I && (22.11)\\
&= |X|^{-1}(q_i(0)A_0 + q_i(1)A_1 + \cdots + q_i(0)A_D) \circ I && (22.12)\\
&= |X|^{-1}q_i(0)I && (22.13)\\
&= |X|^{-1}m_i(E_0 + E_1 + \cdots + E_D). && (22.14)
\end{aligned}$$

This proves the assertions.

□

Definition 22.1. Let $Y = (X, \{R_i\}_{0 \le i \le D})$ be a commutative scheme.

Y is Q-polynomial with respect to ordering $E_0, E_1, \dots, E_D$ of primitive idempotents, if

$$q_{ij}^h \begin{cases} = 0 & \text{if one of } h, i, j \text{ is greater than the sum of the other two,} \\ \neq 0 & \text{if one of } h, i, j \text{ is equal to the sum of the other two.} \end{cases}$$

In this case, set

$$c_i^* = q_{1,i-1}^i,\ a_i^* = q_{1,i}^i,\ b_i^* = q_{1,i+1}^i \quad (0 \le i \le D),\ (c_0^* = b_D^* = 0).$$

Observe: Q-polynomial $\to Y$ is symmetric.

Suppose $i \neq \hat{i}$ for some i. Then, by the condition in Definition 22.1,

$$0 = q_{i\hat{i}}^0 = m_i\ (\neq 0)$$

by Lemma 22.1 (iv). This is a contradiction.

Hence, $E_i{}^\top = E_{\hat{i}} = E_i$ for all i.

Therefore, M is symmetric and Y is symmetric.

Observe: If Y is Q-polynomial,

$$c_i^* + a_i^* + b_i^* = m_1 \quad (0 \le i \le D)$$

(just as $c_i + a_i + b_i = k$ for P-polynomial.)

By Lemma 22.1 (iv),

$$m_1 = q_{10}^i + q_{11}^i + \cdots + q_{1,i-1}^i + q_{1i}^i + q_{1,i+1}^i + \cdots$$

and $q_{10}^i = q_{11}^i = 0$, $q_{1,i-1}^i = c_i^*$, $q_{1i}^i = a_i^*$, and $q_{1,i+1}^i = b_i^*$.

Lemma 22.2. *Assume $Y = (X, \{R_i\}_{0 \le i \le D})$ is a symmetric scheme. Pick $x \in X$, and set $E_i^* \equiv E_i^*(x)$, $A^* \equiv A^*(x)$. Then the following are equivalent.*

(i) Γ *is Q-polynomial with respect to $E_0, \dots, E_D$.*

(ii) *The condition*

$$q_{1j}^h \begin{cases} = 0 & \text{if } |h-j| > 0 \\ \neq 0 & \text{if } |h-j| = 1. \end{cases} \quad (0 \le h, j \le D).$$

(iii) *There exists $f_i^* \in \mathbb{C}[\lambda]$, $\deg f_i^* = i$, and*

$$A_i^* = f_i^*(A_1^*) \quad (0 \le i \le D).$$

(iv) *$E_0^*V, \dots, E_D^*V$ are maximal eigenspaces of A_1^*, and*

$$E_i A_1^* E_j = O \quad \textit{if } |i-j| > 0, \quad (0 \le i, j \le D).$$

(Compare (iv) with the definition of Q-polynomial in Definition 6.2.)

Proof.

$(i) \to (ii)$ Clear.

$(ii) \to (iii)$ $A_0^* = I$,

$$A_i^* A_j^* = \sum_{h=0}^{D} q_{ij}^h A_h^*, \tag{22.15}$$

$$A_1^* A_j^* = q_{1j}^{j-1} A_{j-1}^* + q_{1j}^{j} A_j^* + q_{1j}^{j+1} A_{j+1}^* \qquad (q_{1j}^{j+1} \neq 0, 1 \leq j \leq D-1). \tag{22.16}$$

Hence A_j^* is a polynomial of degree exactly j in A_1^* by induction on j.

$$\lambda f_j^*(\lambda) = b_{j-1}^* f_{j-1}^*(\lambda) + a_j^* f_j^*(\lambda) + c_{j+1}^* f_{j+1}^*(\lambda) \quad \text{with } c_{j+1}^* \neq 0,$$

and $f_{-1}^* = 0$, $f_0^*(\lambda) = 1$.

$(iii) \to (i)$ Pick i, j, h with $0 \leq i, j, h \leq D$ and $h \geq i + j$. Since

$$m_h q_{ij}^h = m_j q_{ih}^j = m_i q_{hj}^i$$

by Lemma 20.2, it suffices to show that

$$q_{ij}^h \begin{cases} = 0 & \text{if } \ h > i+j \\ \neq 0 & \text{if } \ h = i+j. \end{cases}$$

$$A_i^* A_j^* = \sum_{h=0}^{D} q_{ij}^h A_h^* \tag{22.17}$$

$$f_i^*(A_1) f_j^*(A_1) = \sum_{h=0}^{D} q_{ij}^h f_h^*(A_1^*). \tag{22.18}$$

Hence,

$$f_i^*(\lambda) f_j^*(\lambda) = \sum_{h=0}^{D} q_{ij}^h f_h^*(\lambda).$$

Note that since $A_0^*, A_1^*, \dots, A_D^*$ are linearly independent, $f(A_1^*) = 0$ implies $\deg f > D$.

$$\deg \mathrm{LHS} = i + j \to q_{ij}^{i+j} \neq 0, \ q_{ij}^h = 0, \ \text{if} \ \ h > i + j.$$

$(iii) \to (iv)$ Recall

$$A_1^* = q_1(0) E_0^* + q_1(1) E_1^* + \cdots.$$

Each A_i^* is a polynomial in A_1^*. Then A_1^* generates the dual Bose-Mesner algebra. So, $q_1(0), q_1(1), \dots, q_1(D)$ are distinct.

So, $E_0^*V, \ldots, E_D^*V$ are maximal eigenspaces.

Also, $|i-j| > 1$ implies $q_{11}^j = 0$.

Thus, $E_i A_1^* E_j = 0$ by Lemma 20.3 (ii).

$(iv) \to (ii)$ $q_{1j}^i = 0$ if $|i-j| > 1$, since in this case,

$E_i A_1^* E_j = O$ implies $q_{1j}^i = 0$ by Lemma 20.3 (ii).

Suppose $q_{1j}^{j+1} = 0$ for some j $(0 \leq j \leq D-1)$.

Without loss of generalith, choose j minimum. Then A_h^* is a polynomial of degree h in A_1^* $(0 \leq h \leq j)$, and

$$A_1^* A_j^* - q_{1j}^{j-1} A_{j-1}^* - q_{1j}^j A_j^* = O.$$

the left hand side is a polynomial in A_1^* of degree $j+1$.

Hence, the minimal polynomial of A_1^* has degree less than or equal to $j+1 \leq D$. But A_1^* has $D+1$ distince eigenvalues.

This is a contradiction.

□

Chapter 23

Representation of a Scheme

Monday, March 22, 1993

Theorem 23.1. *Let $Y = (X, \{R_i\}_{0 \le i \le D})$ be a symmetric scheme. (View the standard module V as an algebra of functions from X to $\mathbb{C}$.) Then the following are equivalent.*

(i) Y is Q-polynomial with respect to ordering $E_0, E_1, \ldots, E_D$ of primitive idempotents.

(ii) For all i $(0 \le i \le D)$,

$$E_0V + E_1V + (E_1V)^2 + \cdots + (E_1V)^i = E_0V + E_1V + \cdots + E_iV.$$

Proof.

By Lemma 20.4 (ii), (iii).

$$E_h(E_iV \circ E_jV) = 0 \text{ if and only if } q^h_{ij} = 0 \quad (0 \le i, j, h \le D).$$

$(i) \to (ii)$ By our assumption,

$$q^h_{1j} = 0 \text{ if } |h - j| > 1, \text{ and } q^{j+1}_{1j} \ne 0.$$

So,

$$E_1V \circ E_jV \subseteq E_{j-1}V + E_jV + E_{j+1}V \quad (0 \le j \le D), \tag{23.1}$$

$$E_{j+1}(E_1V \circ E_jV) = E_{j+1}V \quad (0 \le j \le D-1), \tag{23.2}$$

by Lemma 20.4.

Also $E_0V \subseteq \mathrm{Span}(\delta)$, where δ is all 1's vector, i.e., 1 as a function $X \to \mathbb{C}$. So,

$$E_0V \circ E_jV = E_jV \quad (0 \le j \le D). \tag{23.3}$$

Show (ii) by induction on i.

The cases $i = 0, 1$ are trivial.

$i > 1$: $\subseteq$.

$$\begin{aligned} & E_0V + E_1V + (E_1V)^2 + \cdots + (E_1V)^i && (23.4)\\ &= E_0V + E_1V \circ (E_0V + E_1V + \cdots + (E_1V)^{i-1}) && (23.5)\\ &= E_0V + E_1V \circ (E_0V + E_1V + \cdots + E_{i-1}V) && (23.6)\\ &\subseteq E_0V + E_1V + \cdots + E_iV && (23.7)\end{aligned}$$

by (23.1).

$\supseteq$.

Claim. $E_iV \subseteq E_1V \circ E_{i-1}V + E_{i-1}V + E_{i-2}V \quad (2 \le i \le D)$.

Proof of Claim. By (23.2),

$$E_i(E_1V \circ E_{i-1}V) = E_iV.$$

For all $v \in E_iV$, there exists $u \in E_1V \circ E_{i-1}V$ such that $E_iu = v$.

On the other hand, by (23.1),

$$E_1V \circ E_{i-1}V \subseteq E_{i-2}V + E_{i-1}V + E_{i-2}V.$$

So, $u = w + v$, where $w \in E_{i-2}V + E_{i-1}V$. We have,

$$w = u - v \in E_1V \circ E_{i-1}V + E_{i-1}V + E_{i-2}V$$

as desired.

HS MEMO

$$E_iV \circ E_jV = \mathrm{Span}(u \circ v \mid u \in E_iV, v \in E_jV).$$

By claim,

$$\begin{aligned} & E_0V + E_1V + \cdots + E_iV && (23.8)\\ &\subseteq E_0V + E_1V + \cdots + E_{i-1}V + E_1V \circ E_{i-1}V && (23.9)\\ &\subseteq E_0V + E_1V + \cdots + (E_1V)^{i-1} + E_1V(E_0V + E_1V + \cdots + (E_1V)^{i-1}) && (23.10)\\ &\subseteq E_0V + E_1V + \cdots + (E_1V)^{i-1} + (E_1V)^i. && (23.11)\end{aligned}$$

$(ii) \to (i)$

Claim 1. Pick i, j $(0 \le i, j \le D)$ with $j > i + 1$. Then $q^j_{1i} = 0$.

Proof of Claim 1.

$$\begin{aligned}
E_j(E_1 \circ E_j V) &\subseteq E_j(E_1 V \circ (E_0 V + E_1 V + (E_1 V)^2 + \cdots + (E_1 V)^i)) && (23.12)\\
&\subseteq E_j(E_0 V + E_1 V + (E_1 V)^2 + \cdots + (E_1 V)^{i+1}) && (23.13)\\
&= E_j(E_0 V + E_1 V + \cdots + E_{i+1} V) && (23.14)\\
&= 0. && (23.15)
\end{aligned}$$

So $q_{1i}^j = 0$ by Lemma 20.4.

Claim 2. $q_{1i}^{i+1} \neq 0$ $(0 \leq i < D)$.

Proof of Claim 2.

$$\begin{aligned}
&E_0 V + E_1 V + \cdots + E_{i+1} V && (23.16)\\
&\quad = E_0 V + E_1 V + \cdots + (E_1 V)^{i+1} && (23.17)\\
&\quad = E_0 V + E_1 V \circ (E_0 V + E_1 V + \cdots + (E_1 V)^i) && (23.18)\\
&\quad = E_0 V + E_1 V \circ (E_0 V + E_1 V + \cdots + E_i V) && (23.19)\\
&\quad = E_0 V + E_1 V \circ (E_0 V + \cdots + E_i V). && (23.20)
\end{aligned}$$

So,

$$\begin{aligned}
E_{i+1} V &= E_{i+1}(E_1 V \circ (E_0 V + \cdots + E_i V)) && (23.21)\\
&= E_{i+1}(E_1 V \circ E_i V) && (23.22)
\end{aligned}$$

by Claim 1 and Lemma 20.4.

Hence, $q_{1i}^{i+1} \neq 0$ by Lemma 20.4.

□

Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ be a commutative scheme with standard module V.

Definition 23.1. A representation of Y is a pair (ρ, H), where H is a non-zero Hermitean space (with inner product $\langle\ ,\ \rangle$) and $\rho : X \to H$ is a map satisfying the following.

R1. $H = \mathrm{Span}(\rho(x) \mid x \in X)$.

R2. $\langle \rho(x), \rho(y) \rangle$ depends only on i for which $(x, y) \in R_i$ $(x, y \in X)$.

R3. For every $x \in X$ and for all i $(0 \leq i \leq D)$,

$$\sum_{y \in X, (y,x) \in R_i} \rho(y) \in \mathrm{Span}(\rho(x)).$$

Above representation is nondegenerate if $\{\rho(x) \mid x \in X\}$ are distinct.

Example 23.1. $Y = H(D,2)$, $X = \{a_1 \cdots a_D \mid a_i \in \{1,-1\}, 1 \le i \le D\}$. Let $H = \mathbb{C}^D$ and $\langle\ ,\ \rangle$ usual Hermitean dot product.

For a vertex $x = a_1 \cdots a_D \in X$, define

$$\rho(x) = a_1 \cdots a_D \in H.$$

Then, $\mathrm{R1} - \mathrm{R3}$ hold.

HS MEMO

$\mathrm{R1}, \mathrm{R2}$ are obvious. For R3, we may assume that $x = 1 \cdots 1$. Restrict

$$\sum_{y \in X, (y,x) \in R_i} \rho(y)$$

on the first coordinate. Then,

$$-1 \quad \text{appears} \quad \binom{D-1}{i-1} \quad \text{times} \tag{23.23}$$

$$1 \quad \text{appears} \quad \binom{D-1}{i} \quad \text{times.} \tag{23.24}$$

Hence,

$$\sum_{y \in X, (y,x) \in R_i} \rho(y) = \left(\binom{D-1}{i} - \binom{D-1}{i-1} \right) \rho(x).$$

Let (ρ, H) be a representation of arbitrary commutative scheme Y. Set

$$E = (\langle \rho(x), \rho(y) \rangle)_{x,y \in X}$$

Gram matrix of the representation.

Definition 23.2. Representations (ρ, H), (ρ', H') of Y are equivalent, whenever, Gram matrices are related by

$$E' \in \mathrm{Span} E.$$

We do not distinguish between equivalent representations.

Note. Suppose (ρ, H) is a representation of a symmetric scheme Y. Pick $x, y \in X$ with $(x,y) \in R_j$.

Then $(y,x) \in R_j$. So, by R2,

$$\langle \rho(x), \rho(y) \rangle = \langle \rho(y), \rho(x) \rangle = \overline{\langle \rho(x), \rho(y) \rangle},$$

since $\langle\ ,\ \rangle$ is Hermitean.

Hence, the Gram matrix E of ρ is real symmetirc. Without loss of generality, we can view H as a real Euclidean space in this case.

Lemma 23.1. *Let $Y = (X, \{R_i\}_{0 \le i \le D})$ be a commutative scheme and V a standard module.*

Let E_j be any primitive idempotent of Y.

(i) (ρ, H) is a representation of Y, where $H = E_j V$ (with inner product inherited from Y).

$$\rho : X \to H \quad (x \mapsto E_j \hat{x})$$

(i.e., $\rho(x)$ is the x-th column of E_j.)

(ii) $\langle \rho(x), \rho(y) \rangle = |X|^{-1} q_j(i)$, if $(x, y) \in R_i$, $(x, y \in X)$.

(iii) For $0 \le i \le D$ and $x, y \in X$,

$$\sum_{y \in X, (y,x) \in R_i} \rho(y) = p_i(j) \rho(x).$$

(iv) (ρ, H) is nondegenerate if and only if $q_j(i) \ne q_j(0)$ for all i, $(0 \le i \le D)$.

(v) Every representation of Y is equivalent to a representation of the above type for some j $(0 \le j \le D)$, and j is unique.

Proof.

$(i) - (iii)$.

R1: Span(ρX) is the column space of E_j which is equal to H.

R2:

$$\begin{aligned}
\langle \rho(x), \rho(y) \rangle &= \langle E_j \hat{x}, E_j \hat{y} \rangle &\quad (23.25)\\
&= (\overline{E_j \hat{x}})^\top E_j \hat{y} &\quad (23.26)\\
&= \hat{x}^\top \overline{E_j}^\top E_j \hat{y} &\quad (23.27)\\
&= \hat{x}^\top E_j \hat{y} &\quad (23.28)\\
&= (E_j)_{xy}. &\quad (23.29)
\end{aligned}$$

Note that $\overline{E_j}^\top = E_j$ by Lemma 19.1.

Recall

$$E_j = |X|^{-1} (q_j(0) A_0 + \cdots + q_j(D) A_D).$$

So,

$$(E_j)_{xy} = |X|^{-1} q_j(i), \quad \text{where } (x, y) \in R_i.$$

R3: Recall

$$A_i = p_i(0) E_0 + \cdots + p_i(D) E_D.$$

So, $E_jA_i = p_i(j)E_j$, and

$$p_i(j)\rho(x) = p_i(j)E_j\hat{x} = E_jA_i\hat{x} = E_j\sum_{y\in X,(y,x)\in R_i}\hat{y} = \sum_{y\in X,(y,x)\in R_i}\rho(y).$$

Note.

$$A_i\hat{x} = \sum_{y\in X,(x,y)\in R_{i'}}\hat{y}.$$

Pf.

$$\begin{aligned} z \text{ entry of LHS} &= (A_i\hat{x})_z && (23.30)\\ &= \sum_{w\in X}(A_i)_{zw}\hat{x}_w && (23.31)\\ &= (A_i)_{zx} && (23.32)\\ &= \begin{cases} 1 & \text{if } (x,z)\in R_{i'}\\ 0 & \text{else.}\end{cases} && (23.33)\end{aligned}$$

$$\begin{aligned} z \text{ entry of RHS} &= \sum_{y\in X,(x,y)\in R_{i'},z=y} 1 && (23.34)\\ &= \begin{cases} 1 & \text{if } (x,z)\in R_{i'}\\ 0 & \text{else.}\end{cases} && (23.35)\end{aligned}$$

(iv) By (ii),

$$\begin{aligned} \|\rho(x)\|^2 &= \langle\rho(x),\rho(y)\rangle && (23.36)\\ &= |X|^{-1}q_j(0) && (23.37)\\ &= |X|^{-1}m_j, && (23.38)\end{aligned}$$

as $m_j = \dim E_jV$, and is independent of $x\in X$.

Pick distinct $x, y\in X$ such that $(x,y)\in R_i$ with $i\neq 0$.

Then,

$$\begin{aligned} \rho(x)=\rho(y) &\Leftrightarrow \langle\rho(x),\rho(y)\rangle = \|\rho(x)\|^2 = |X|^{-1}q_j(0) && (23.39)\\ &\Leftrightarrow |X|^{-1}q_j(i) = |X|^{-1}q_j(0) && (23.40)\\ &\Leftrightarrow q_j(i) = q_j(0). && (23.41)\end{aligned}$$

Hence, we have (iv). To be continued.

□

Chapter 24

Balanced Conditions, I

Wednesday, March 23, 1993

No Class on Friday (another conference).

Proof of Lemma 23.1 continued. Let E_j be a primitive idempotent, $H = E_j V$ and

$$\rho : X \to H \quad (x \mapsto E_j \hat{x}).$$

(v) Every representation (ρ, H) of Y is equivalent to a representation of above type, for some j $(0 \leq j \leq D)$ and j is unique.

Let $E := (\langle \rho(x), \rho(y) \rangle)_{x,y \in X}$.

By R2,

$$E = \sum_{i=0}^{D} \sigma_i A_i, \quad \text{some } \sigma_0, \sigma_1, \dots, \sigma_D \in \mathbb{C}.$$

Hence, E belongs to the Bose-Mesner algebra M of Y.

We want to show that E is a scalar multiple of a primitive idempotent.

Fix $x \in X$ and fix i $(0 \leq i \leq D)$.

By R3,

$$\sum_{y \in X, (y,x) \in R_i} \rho(y) = \alpha \rho(x), \quad \text{some } \ \alpha \in \mathbb{C}. \tag{24.1}$$

So,

$$k_i \overline{\sigma_i} = \left\langle \sum_{y \in X, (y,x) \in R_i} \rho(y), \rho(x) \right\rangle = \bar{\alpha} \langle \rho(x), \rho(x) \rangle = \bar{\alpha} \sigma_0.$$

Hence, α is independent of x. In maatrix form (24.1) becomes

$$E A_i \hat{x} = \alpha E \hat{x}.$$

HS MEMO

$$Eu = Ev \Leftrightarrow \langle z, Eu\rangle = \langle z, Ev\rangle \text{ for all } z \in X \Leftrightarrow (Eu)_z = (Ev)_z \text{ for all } z \in X.$$

$$(EA_i\hat{x})_z = \left\langle \rho(z), \sum_{y\in X,(y,x)\in R_i} \rho(y) \right\rangle \tag{24.2}$$
$$= \alpha\langle \rho(z), \rho(x)\rangle \tag{24.3}$$
$$= (\alpha E\hat{x})_z. \tag{24.4}$$

Hence,

$$EA_i\hat{x} = \alpha E\hat{x}.$$

Since x is arbitrary,

$$EA_i = \alpha E.$$

So,

$$EA_i \in \mathrm{Span}E \quad \text{and} \quad EM = \mathrm{Span}E.$$

We have $E \in \mathrm{E_j}$ for unique j $(0 \le j \le D)$. □

HS MEMO

$$E = \tau_0 E_0 + \cdots + \tau_D E_D,\ \tau_j \in \mathbb{C} \quad (0 \le j \le D).$$

And, at least one of τ_j is nonzero, and

$$\tau_j E_j = EE_j \in \mathrm{Span}E.$$

So,

$$\tau_j E_j = E$$

as $E_0, \ldots, E_D$ are linearly independent.

Let $Y = (X, \{R_i\}_{0\le i\le D})$ be a symmetric scheme, and let E be a primitive idempotent.

Definition 24.1. Y is Q-polynomial with respect to E, if and only if Y is Q-polynomial with respect to some ordering $E_0, E_1, \ldots, E_D$ of primitive idempotents, where $E_0 = |X|^{-1}J$, and $E_1 = E$.

Theorem 24.1. *Assume $Y = (X, \{R_i\}_{0\le i\le D})$ is P-polynomial (i.e., (X, R_1) is distance-regular). Let E be any primitive idempotent of Y. Let (ρ, H) be the corresponding representation.*

(i) *The following are equivalent.*

(ia) *Y is Q-polyonimial with respect to E.*

(ib) (ρ, H) *is nondegenerate and for all* $x, y \in X$*, and for all* i, j $(0 \leq i, j \leq D)$,

$$\sum_{z \in X, (x,z) \in R_i, (y,z) \in R_j} \rho(z) - \sum_{z' \in X, (x,z') \in R_j, (y,z') \in R_i} \rho(z') \in \operatorname{Span}(\rho(x) - \rho(y)).$$

(ic) (ρ, H) *is nondegenerate and for all* $x, y \in X$,

$$\sum_{z \in X, (x,z) \in R_1, (y,z) \in R_2} \rho(z) - \sum_{z' \in X, (x,z') \in R_2, (y,z') \in R_1} \rho(z') \in \operatorname{Span}(\rho(x) - \rho(y)).$$

(ii) *Write*

$$E = |X|^{-1} \sum_{j=0}^{D} \theta_j^* A_j,$$

and suppose $(ia) - (ic)$ *hold. Then the coefficient in* (ib) *is*

$$p_{ij}^h \frac{\theta_i^* - \theta_j^*}{\theta_0^* - \theta_h^*} \quad (1 \leq h \leq D, 0 \leq i, j \leq D).$$

Proof.

$(ia) \to (ib)$ Without loss of generality, assume $E \equiv E_1$, and Y is Q-polynomial with respect to E.

Then by Lemma 22.2, $\theta_0^*, \dots, \theta_D^*$ are distinct. So $\theta_h^* \neq \theta_0^*$ for all $h \in \{1, 2 \dots, D\}$, and (ρ, H) is nondegenerate.

Fix $x \in X$, write $E_i^* \equiv E_i^*(x)$, $A_i^* \equiv A_i^*(x)$, $A^* \equiv A_1^*$.

Let M be the Bose-Mesner algebra. Set

$$L = \{mA^*n - nA^*m \mid m, n \in M\}.$$

Claim 1. $\dim L \leq D$.

Proof of Claim 1.

$$\begin{aligned} L &= \operatorname{Span}(E_i A^* E_j - E_j A^* E_i \mid 0 \leq i < j \leq D) && (24.5) \\ &= \operatorname{Span}(E_i A^* E_{i+1} - E_{i+1} A^* E_i \mid 0 \leq i \leq D-1). && (24.6) \end{aligned}$$

Since $E_i A^* E_j = 0$ if $q_{ij}^1 = 0$ by Lemma 20.2 and Lemma 20.3, and this occurs if $|i - j| > 1$ by Q-polynomial property.

Hence, $\dim L \leq D$.

Claim 2. (i) $\{A^*A_h - A_hA^* \mid 1 \le h \le D\}$ is a basis for L. In particular,

(ii) there exist $r^h_{ij} \in \mathbb{C}$ $(1 \le h \le D, 0 \le i, j \le D)$ such that

$$A_iA^*A_j - A_jA^*A_i = \sum_{h=1}^{D} r^h_{ij}(A^*A_h - A_hA^*).$$

Proof of Claim 2.

(i) The column x of $A^*A_h - A_hA^*$ is a nonzero scalar $\theta^*_h - \theta^*_0$ times the column x of A_h.

HS MEMO

$$((A^*A_h - A_hA^*)\hat{x})_y = E_{xy}(A_h)_{yx} - (A_h)_{yx}E_{xx} = (\theta^*_h - \theta^*_0)(A_h)_{yz}.$$

Also the column x of $A_0, A_1, \dots, A_D$ are linearly independent.

Hence, the matrices given are linearly independent.

They are in L by construction, so they form a basis for L by Claim 1.

(ii) This is immediate since

$$A_iA^*A_j - A_jA^*A_i \in L, \quad \text{for all } i, j.$$

Claim 3.

$$r^\ell_{ij} = p^\ell_{ij}\left(\frac{\theta^*_i - \theta^*_j}{\theta^*_0 - \theta^*_\ell}\right) \quad (1 \le \ell \le D, 0 \le i, j \le D).$$

Proof of Claim 3. Fix i, j,

$$A_iA^*A_j - A_jA^*A_i - \sum_{h=1}^{D} r^h_{ij}(A^*A_h - A_hA^*) = 0.$$

Pick ℓ $(1 \le \ell \le D)$. Pick $y \in X$ such that $(x, y) \in R_\ell$.

$$\begin{aligned}(A_iA^*A_j)_{xy} &= \sum_{z \in X}(A_i)_{xz}(A^*)_{zz}(A_j)_{zy} && (24.7)\\ &= \sum_{z \in X, (x,z) \in R_i, (y,z) \in R_j} (A^*)_{zz} && (24.8)\\ &= |X|^{-1}p^\ell_{ij}\theta^*_i. && (24.9)\end{aligned}$$

Similarly,

$$(A_jA^*A_i)_{xy} = |X|^{-1}p^\ell_{ij}\theta^*_j.$$

$$(A^*A_h - A_hA^*)_{xy} = (A_0A^*A_h - A_hA^*A_0)_{xy} \tag{24.10}$$
$$= |X|^{-1}p^{\ell}_{0h}(\theta^*_0 - \theta^*_h) \tag{24.11}$$
$$= \begin{cases} 0 & \text{if } \ell \neq h \\ |X|^{-1}(\theta^*_0 - \theta^*_h) & \text{if } \ell = h. \end{cases} \tag{24.12}$$

Hence,

$$\sum_{h=1}^{D} r^h_{ij}(A^*A_h - A_hA^*)_{xy} = |X|^{-1}r^{\ell}_{ij}(\theta^*_0 - \theta^*_{\ell}).$$

Comparing terms, we have

$$p^{\ell}_{ij}(\theta^*_i - \theta^*_j) - r^{\ell}_{ij}(\theta^*_0 - \theta^*_{\ell}) = 0.$$

Claim 4. For all h ($1 \leq h \leq D$), for all i, j ($0 \leq i, j \leq D$), for all $w, y \in X$, $(w, y) \in R_h$,

$$\sum_{z \in X, (w,z) \in R_i, (y,z) \in R_j} \rho(z) - \sum_{z' \in X, (w,z') \in R_j, (y,z) \in R_i} \rho(z') - r^h_{ij}(\rho(w) - \rho(y)) = 0. \tag{24.13}$$

Proof of Claim 4. Set $L = \langle \text{LHS of (24.13)}, \rho(x) \rangle$ It suffices to show that $L = 0$. Note that since x is arbitrary, if LHS of (24.13) is zero.

$$L = \sum_{z \in X, (w,z) \in R_i, (y,z) \in R_j} \langle \rho(z), \rho(x) \rangle - \sum_{z' \in X, (w,z') \in R_j, (y,z) \in R_i} \langle \rho(z'), \rho(x) \rangle \tag{24.14}$$
$$- r^h_{ij} \langle \rho(w) - \rho(y), \rho(x) \rangle \tag{24.15}$$
$$= |X|^{-1}(A_iA^*A_j)_{wy} - |X|^{-1}(A_jA^*A_i)_{wy} - |X|^{-1} \sum_{\ell=1}^{D} r^{\ell}_{ij}(A^*A_{\ell} - A_{\ell}A^*)_{wy} \tag{24.16}$$
$$= |X|^{-1}\text{times } wy \text{ entry of a matrix known to be zero by Claim 2} \tag{24.17}$$
$$= 0. \tag{24.18}$$

Thus we have the claim.

□

HS MEMO

$$|X|^{-1} \sum_{\ell=1}^{D} r^{\ell}_{ij}(A^*A_{\ell} - A_{\ell}A^*)_{wy} = |X|^{-1}r^h_{ij}(A^*A_h - A_hA^*)_{wy} \tag{24.19}$$
$$= r^h_{ij}(\langle \rho(x), \rho(w) \rangle - \langle \rho(x), \rho(y) \rangle) \tag{24.20}$$

Chapter 25

Balanced Conditions, II

Monday, March 29, 1993

Proof of Theorem 24.1 continued.

$(ib) \to (ic)$ Obvious.

$(ic) \to (ia)$ Without loss of generality, we may assume $D \geq 3$, else trivial.

HS MEMO

The case $D = 2$ should be treated somewhere, but the assumption $D \geq 3$ is not used.

Fix $w \in X$, and write $E_i^* \equiv E_i^*(w)$, $A_i^* \equiv A_i^*(w)$, $A^* \equiv A_1^*$, and A_i, i-th distance matrix. Set

$$E \equiv E_1 = |X|^{-1} \sum_{i=0}^{D} \theta_i^* A_i.$$

Since (ρ, H) is nondegenerate,

$$\theta_0^* \neq \theta_h^* \text{ for all } h \in \{1, 2, \ldots, D\}$$

See Lemma 23.1 (iv).

Claim 1. Pick h $(1 \leq h \leq D)$, and x, y with $(x, y) \in R_h$. Then

$$\sum_{z \in X, (x,z) \in R_1, (y,z) \in R_2} \rho(z) - \sum_{z' \in X, (x,z') \in R_2, (y,z') \in R_1} \rho(z') = r_{12}^h (\rho(x) - \rho(y)),$$

where

$$r_{12}^h = p_{12}^h \frac{\theta_1^* - \theta_2^*}{\theta_0^* - \theta_h^*}.$$

Proof of Claim 1. By our assumption,

$$\sum_{z\in X,(x,z)\in R_1,(y,z)\in R_2} \rho(z) - \sum_{z'\in X,(x,z')\in R_2,(y,z')\in R_1} \rho(z') = \alpha(\rho(x)-\rho(y)).$$

Hence,

$$|X|^{-1}p^h_{12}(\theta^*_1-\theta^*_2) = \left\langle \sum_{z\in X,(x,z)\in R_1,(y,z)\in R_2} \rho(z) - \sum_{z'\in X,(x,z')\in R_2,(y,z')\in R_1} \rho(z'), \rho(x)\right\rangle \tag{25.1}$$

$$= \alpha\langle\rho(x)-\rho(y),\rho(x)\rangle \tag{25.2}$$

$$= \alpha|X|^{-1}(\theta^*_0-\theta^*_h). \tag{25.3}$$

We have

$$\alpha = p^h_{12}\frac{\theta^*_1-\theta^*_2}{\theta^*_0-\theta^*_h}.$$

Claim 2. $A_1A^*A_2 - A_2A^*A_1 = \sum_{h=1}^{D} r^h_{12}(A^*A_h - A_hA^*)$.

Proof of Claim 2. The xy entry of the LHS − RHS is

$$|X|\left\langle \sum_{z\in X,(x,z)\in R_1,(y,z)\in R_2} \rho(z) - \sum_{z'\in X,(x,z')\in R_2,(y,z')\in R_1} \rho(z') - r^h_{12}(\rho(x)-\rho(y)), \rho(w)\right\rangle,$$

where $(x,y)\in R_h$, $h=1,2,\dots,D$, and the xy entry of the LHS − RHS is 0 if $x=y$.

But the vector on the left in the above inner product is 0 by Claim 1, so the inner product is 0.

Thus, the xy entry of the LHS − RHS is always 0, and we have Claim 2.

Claim 3. $A^*A_3 - A_3A^* \in \mathrm{Span}(AA^*A_2 - A_2A^*A, A^*A_2 - A_2A^*, A^*A - AA^*)$.

Proof of Claim 3. Since $p^h_{12}=0$, if $h>3$, and $p^h_{12}\neq 0$, if $h=3$, we have $r^h_{12}=0$ if $h>0$, and $r^h_{12}\neq 0$, if $h=3$. Note that $\theta^*_1\neq\theta^*_2$.

Now we are done by Claim 2.

Claim 4. There exist $\beta,\gamma,\delta\in\mathbb{R}$ such that

$$0 = [A, A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^*+A^*A) - \delta A^*] \tag{25.4}$$

$$= A^3A^* - A^*A^3 - (\beta+1)(A^2A^*A - AA^*A^2) - \gamma(A^2A^* - A^*A^2) - \delta(AA^* - A^*A). \tag{25.5}$$

Proof of Claim 4. There exists $f_i\in\mathbb{R}[\lambda]$, $\deg f_i = i$ such that $A_i = f_i(A_1)$.

Writing A_2, A_3 as polynomials in A in Claim 3 and simplifying, we find

$$A^3A^* - A^*A^3 \in \mathrm{Span}(A^2A^*A - AA^*A^2, A^2A^* - A^*A^2, AA^* - A^*A).$$

HS MEMO

Let $A_3 = \beta_3 A^3 + \beta_2 A^2 + \beta_1 A + \beta_0 I$ with $\beta_3 \neq 0$, and $A_2 = \gamma_2 A^2 + \gamma_1 A + \gamma_0 I$, with $\gamma_2 \neq 0$. Then

$$A^*A_3 - A_3A^* = A^*(\beta_3 A^3 + \beta_2 A^2 + \beta_1 A + \beta_0 I) - (\beta_3 A^3 + \beta_2 A^2 + \beta_1 A + \beta_0 I)A^*. \tag{25.6}$$

$$A^3A^* - A^*A^3 \in \mathrm{Span}(A^*A_3 - A_3A^*, A^2A^* - A^*A^2, AA^* - A^*A) \tag{25.7}$$

$$\subseteq \mathrm{Span}(AA^*A_2 - A_2A^*A, A^*A_2 - A_2A^*, A^2A^* - A^*A^2, AA^* - A^*A) \tag{25.8}$$

$$A^*A_2 - A_2A^* = A^*(\gamma_2 A^2 + \gamma_1 A + \gamma_0 I) - (\gamma_2 A^2 + \gamma_1 A + \gamma_0 I)A^* \tag{25.9}$$

$$AA^*A_2 - A_2A^*A = AA^*(\gamma_2 A^2 + \gamma_1 A + \gamma_0 I) - (\gamma_2 A^2 + \gamma_1 A + \gamma_0 I)A^*A \tag{25.10}$$

$$A^*A_2 - A_2A^* \in \mathrm{Span}(A^2A^* - A^*A^2, AA^* - AA^*) \tag{25.11}$$

$$AA^*A_2 - A_2A^*A \in \mathrm{Span}(A^2A^*A - AA^*A^2, AA^* - AA^*) \tag{25.12}$$

$$A^3A^* - A^*A^3 \in \mathrm{Span}(A^2A^*A - AA^*A^2, A^2A^* - A^*A^2, AA^* - A^*A). \tag{25.13}$$

Hence, we can find δ, γ, δ satisfying

$$0 = A^3A^* - A^*A^3 - (\beta+1)(A^2A^*A - AA^*A^2) - \gamma(A^2A^* - A^*A^2) - \delta(AA^* - A^*A).$$

On the other hand,

$$[A, A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \delta A^*] \tag{25.14}$$

$$= A^3A^* - A^2A^*A - \beta A^2A^*A + \beta AA^*A^2 + AA^*A^2 - A^*A^3 \tag{25.15}$$

$$- \gamma A^2A^* - \gamma AA^*A + \gamma AA^*A + \gamma A^*A^2 - \delta AA^* + \delta A^*A \tag{25.16}$$

$$= A^3A^* - A^*A^3 - (\beta+1)(A^2A^*A - AA^*A^2) - \gamma(A^2A^* - A^*A^2) - \delta(AA^* - A^*A). \tag{25.17}$$

Thus we have (i) and (ii).

Define a diagram D_E on nodes $0, 1, \dots, D$.

Connect distinct nodes , by undirected arc if $q^1_{ij} \neq 0$. (Note $q^1_{ij} = q^1_{ji}$).

Since $q^1_{0j} = \delta_{1j}$, the 0-node is adjacent to the 1-node and no other node.

Y is Q-polynomial with respect to E if and only if D_E is a path.

Claim 5. D_E is connected.

Proof of Claim 5. Suppose there exists $\Delta \subseteq \{0, 1, \dots, D\}$ such that i, j not connected for every $i \in \Delta$ and $j \in \{0, 1, \dots, D\} \quad \Delta$.

Set
$$f = \sum_{i \in \Delta} E_i.$$

Observe

$$\begin{aligned} fA^* &= \sum_{i\in\Delta} E_i A^* \left(\sum_{j=0}^{D} E_j \right) && (25.18)\\ &= \sum_{i\in\Delta, j\in\Delta} E_i A^* E_j \quad (\text{since } E_i A^* E_j = O \text{ if } q^1_{ij} = 0) && (25.19)\\ &= fA^* f. && (25.20) \end{aligned}$$

Also, $A^* f = fA^* f$.

Hence, f commutes with A^*.

But f is an element of the Bose-Mesner algebra

$$f = \sum_{i=0}^{D} \alpha_i A_i \quad \text{for some } \alpha_0, \dots, \alpha_D \in \mathbb{C}.$$

We have
$$0 = fA^* - A^* f = \sum_{i=1}^{D} \alpha_i (A_i A^* - A^* A_i).$$

But $\{A_h A^* - A^* A_h \mid 1 \le h \le D\}$ are linearly independent. (The column w of $A_h A^* - A^* A_h$ is $\theta^*_h - \theta^*_0$ times the column w of A_h.)

Hence, $\alpha_1 = \cdots = \alpha_D = 0$, and $f = \alpha_0 I$. Since $f^2 = f$, α_0 or 1.

If $\alpha_0 = 0$, $f = O$ and $\Delta = \emptyset$.

If $\alpha_0 = 1$, $f = I$ and $\Delta = \{0, 1, \dots, D\}$.

This proves Claim 5.

$\square$

HS MEMO

Claim 5 proves the following in general.

Let $Y = (X, \{R_i\}_{0 \le i \le D})$ be a symmetric association scheme. Fix a vertex $x \in X$, and let
$$E = \frac{1}{|X|} \sum_{j=0}^{D} \theta^*_j A_j \quad (\theta^*_j = q_1(j) \ \text{ if } E = E_1)$$

be a primitive idempotent and $E^*_j \equiv E^*_j(x)$.

$$A^* = \sum_{j=0}^{D} \theta^*_j E^*_j.$$

If $\theta_0 = \theta_h^*$, $h = 1, \dots, D$, then the following hold.

(i) $\{A_h A^* - A^* A_h \mid 1 \le h \le D\}$ are linearly independent.

(ii) The diagram D_E on nodes $0, 1, \dots, D$ defined by

$$i \sim j \Leftrightarrow E(E_i \circ E_j) \neq O$$

is connected.

(iii) $C_M(A^*) = \{L \in M \mid LA^* = A^*L\} = \mathrm{Span}(I)$.

Proof.

(i) The column x of $A_h A^* - A^*(A_h)$ is $\theta_0^* - \theta_h^*$ times the column x of A_h.

(iii) $0 = \left[\sum_{h=0}^{D} \alpha_h A_h, A^*\right] = \sum_{h=1}^{D} \alpha_h (A_h A^* - A^* A_h)$. Hence, $\alpha_0 = \cdots = \alpha_D = 0$.

(ii) Δ is a connected component. Let $f = \sum_{i \in \Delta} E_i$, then $f \in C_M(A^*)$.

Let $Y = (X, \{R_i\}_{0 \le i \le 2})$ be a symmetric association scheme with $D = 2$. Let

$$E = \frac{1}{|X|} \sum_{j=0}^{2} \theta_j^* A_j$$

be a primitive idempotent.

Suppose $\theta_0^* \neq \theta_1^*, \theta_2^*$. Then Y is Q-polynomial with respect to E.

Proof. By the previous lemma, D_E is connected.

Note. It seems $\theta_1^* \neq \theta_2^*$ is necessary. Clarify the condition $\theta_1^* = \theta_2^*$.

Terwilliger claims that $\theta_1^* = \theta_2^*$ does not occur under the assumption (ic). (March 7, 1995)

Chapter 26

Representation Diagrams

Wednesday, March 31, 1993

Proof of Theorem 24.1 continued. Assume $Y = (X, \{R_i\}_{0\le i\le D})$ is P-polynomial. Let E be a primitive idempotent of Y such that the corresponding representation (ρ, H) is nondegenerate.

Show for all $x, y \in X$,

$$\sum_{z\in X,(x,z)\in R_1,(y,z)\in R_2} \rho(z) - \sum_{z'\in X,(x,z')\in R_2,(y,z')\in R_1} \rho(z') \in \operatorname{Span}(\rho(x)-\rho(y))$$

implies that Y is Q-polynomial with respect to E.

Define a diagram D_E on nodes $0, 1, \dots, D$, for $i \neq j$,

$$i \frown j \leftrightarrow q^1_{ij} \neq 0$$

by setting $E = E_1$.

We showed that $0 \frown j \leftrightarrow j = 1$ $(1 \le j \le D)$ and D_E is connected.

Now it is sufficient to show the following.

Claim 6. Let i be a node in D_E. Then i is adjacent to at most 2 arcs.

Proof of Claim 6. Suppose the node j is adjacent to i in D_E. By Claim 4,

$$\begin{aligned} 0 &= E_i(A^3A^* - A^*A^3 - (\beta+1)(A^2A^*A - AA^*A^2) - \gamma(A^2A^* - A^*A) - \delta(AA^* - A^*A))E_j \quad (26.1)\\ &= E_iA^*E_j(\theta_i^3 - \theta_j^3 - (\beta+1)(\theta^2\theta_j - \theta_i\theta_j^2) - \gamma(\theta_i^2 - \theta_j^2) - \delta(\theta_i - \theta_j)) \quad (26.2)\\ &= E_iA^*A_j(\theta_i - \theta_j)p(\theta_i, \theta_j), \quad (26.3)\end{aligned}$$

where

$$p(s,t) = s^2 - \beta st + t^2 - \gamma(s+t) - \delta.$$

HS MEMO

$$
\begin{aligned}
& (\theta_i-\theta_j)(\theta_i^2-\beta\theta_i\theta_j+\theta_j^2-\gamma(\theta_i+\theta_j)-\delta) && (26.4)\\
= \quad & \theta_i^3-\theta_j^3-(\beta+1)(\theta^2\theta_j-\theta_i\theta_j^2)-\gamma(\theta_i^2-\theta_j^2)-\delta(\theta_i-\theta_j) && (26.5)
\end{aligned}
$$

Since i is adjacent to j, $q_{ij}^1 \neq 0$ and

$$E_i A^* E_j \neq O$$

by Lemma 20.3 (ii). Since Y is P-polynomial,

$$\theta_i \neq \theta_j \quad \text{if } \; i \neq j.$$

Hence $p(\theta_i, \theta_j) = 0$. But p is quadratic in t. So $p(\theta_i, t) = 0$ has at most two solutions for θ_j.

Now D_E is a pth, and Γ is Q-polynomial with respect to E.

This proves Theorem 24.1. □

Corollary 26.1. *Assume $Y = (X, \{R_i\}_{0\le i\le D})$ is P-polynomial, and Q-polynomial with respect to a primitive idempotent*

$$E = \frac{1}{|X|}\sum_{i=0}^{D}\theta_i^* A_i.$$

Then,

$$\beta = \frac{\theta_i^*-\theta_{i+1}^*+\theta_{i+2}^*-\theta_{i+3}^*}{\theta_{i+1}^*-\theta_{i+2}^*}$$

is independent of i $(0 \le i \le D-3)$.

Proof. Fix i. Without loss of generality, $D \ge 3$, else vacuous.

Pick $x, y \in X$ with $(x, y) \in R_3$.

Let (ρ, H) be the representation for E.

$$\sum_{z\in X,(x,z)\in R_1,(y,z)\in R_2}\rho(z) - \sum_{z'\in X,(x,z')\in R_2,(y,z')\in R_1}\rho(z') = \frac{p_{12}^3(\theta_1^*-\theta_2^*)}{\theta_0^*-\theta_3^*}(\rho(x)-\rho(y)), \tag{26.6}$$

and $p_{12}^3 = c_3$.

Since $p_{i,i+3}^3 \neq 0$, there exists $w \in X$ such that $(x, w) \in R_{i+3}$, $(y, w) \in R_i$.

Take inner product of (26.6) with $\rho(w)$. We have

$$P_{12}^3(x,y) \subseteq P_{1,i+2}^{i+3}(x,w)\cap P_{2,i+2}^{i}(y,w) \tag{26.7}$$

$$P_{21}^3(x,y) \subseteq P_{2,i+1}^{i+3}(x,w)\cap P_{2,i+1}^{i}(y,w). \tag{26.8}$$

Hence,

$$\left\langle \sum_{z\in X,(x,z)\in R_1,(y,z)\in R_2} \rho(z) - \sum_{z'\in X,(x,z')\in R_2,(y,z')\in R_1} \rho(z'), \rho(w) \right\rangle = |X|^{-1}c_3(\theta^*_{i+2}-\theta^*_{i+1}),$$

$$\left\langle \frac{c_3(\theta^*_1-\theta^*_2)}{\theta^*_0-\theta^*_3}(\rho(x)-\rho(y)), \rho(w) \right\rangle = \frac{c_3(\theta^*_1-\theta^*_2)}{\theta^*_0-\theta^*_3}|X|^{-1}(\theta^*_{i+3}-\theta^*_{i+1}).$$

We have,

$$\sigma = \frac{\theta^*_{i+1}-\theta^*_{i+2}}{\theta^*_i-\theta^*_{i+3}} = \frac{\theta^*_1-\theta^*_2}{\theta^*_0-\theta^*_3}.$$

HS MEMO

Note that since Y is P and Q with respect to A_1 and E_1, $\theta^*_0, \theta^*_1, \dots, \theta^*_D$, $\theta_0, \theta_1, \dots, \theta_D$ are all distinct.

So

$$\beta = \frac{1}{\sigma} - 1 = \frac{\theta^*_i - \theta^*_{i+1} + \theta^*_{i+2} - \theta^*_{i+3}}{\theta^*_{i+1}-\theta^*_{i+2}} = \frac{\theta^*_0 - \theta^*_1 + \theta^*_2 - \theta^*_3}{\theta^*_1-\theta^*_2}.$$

We have the assertion. □

Given the intersection number of a distance-regular graph Γ. The following two lemmas give an efficient method to determine if Γ is Q-polynomial with respect to some primitive idempotent.

Lemma 26.1. *Let Γ be a distance-regular graph of diameter $D \geq 1$. Pick $\theta, \theta^*_0, \theta^*_1, \dots, \theta^*_D \in \mathbb{R}$ such that $\theta^*_0 \neq 0$, and set*

$$E = \frac{1}{|X|}\sum_{i=0}^{D} \theta^*_i A_i.$$

(i) The following are equivalent.

(ia) θ is an eigenvalue of Γ, and E is a corresponding primitive idempotent.

(ib)

$$\begin{pmatrix} a_0 & b_0 & 0 & \cdots & \cdots & 0 \\ c_1 & a_1 & b_1 & 0 & \cdots & 0 \\ 0 & c_2 & a_2 & b_2 & \ddots & \vdots \\ \cdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & c_{D-1} & a_{D-1} & b_{D-1} \\ 0 & \cdots & \cdots & 0 & c_D & a_D \end{pmatrix} \begin{pmatrix} \theta^*_0 \\ \theta^*_1 \\ \vdots \\ \vdots \\ \vdots \\ \theta^*_D \end{pmatrix} = \theta \cdot \begin{pmatrix} \theta^*_0 \\ \theta^*_1 \\ \vdots \\ \vdots \\ \vdots \\ \theta^*_D \end{pmatrix},$$

and $\theta^*_0 = \mathrm{rank} E$.

(ii) Suppose (ia), (ib) hold. Then,

$$\frac{\theta_1^*}{\theta_0^*}, \ldots, \frac{\theta_D^*}{\theta_0^*}$$

can be computed from θ using

$$\frac{\theta_i^*}{\theta_0^*} = \frac{p_i(\theta)}{kb_1 \cdots b_{i-1}} \quad (1 \le i \le D),$$

where $p_0 = 1$, $p_1(\lambda) = \lambda$, and

$$\lambda p_i(\lambda) = p_{i+1}(\lambda) + a_i p_i(\lambda) + b_{i-1} c_i p_{i-1}(\lambda) \quad (0 \le i \le D).$$

Proof.

(i) We have

$$\begin{aligned}
(ia) &\leftrightarrow (A - \theta I)E = 0 \text{ and } E^2 = E && (26.9)\\
&\leftrightarrow 0 = \sum_{i=0}^{D} (A - \theta I)\theta_i^* A_i \text{ and } \mathrm{rank}E = \mathrm{trace}E = \theta_0^* && (26.10)\\
&\quad = \sum_{i=0}^{D} \theta_i^* (c_{i+1}A_{i+1} + a_i A_i + b_{i-1}A_{i-1} - \theta A_i) && (26.11)\\
&\quad = \sum_{j=0}^{D} A_j (c_j \theta_{j-1}^* + a_j \theta_j^* + b_j \theta_{j+1}^* - \theta\theta_j^*) && (26.12)\\
&\leftrightarrow c_j\theta_{j-1}^* + a_j\theta_j^* + b_j\theta_{j+1}^* = \theta\theta_j^* \ (0 \le j \le D) \text{ and } \mathrm{rank}E = \theta_0^* && (26.13)\\
&\leftrightarrow (ib). && (26.14)
\end{aligned}$$

HS MEMO

The first $\leftrightarrow$. $\rightarrow$ is clear.

$\leftarrow$: By the first condition, $AE = \theta E$. So E is a scalar multiple of the primitive idempotent corresponding to θ. Hence, $\mathrm{rank}E = \mathrm{trace}E$ implies E is the primitive idempotent.

(ii) We prove by induction on i.

$i = 0$ is trivial.

$i = 1$: Set $j = 0$ above $c_0 = 0, a_0 = 0, b_0 = k$. We have

$$k\theta_1^* = \theta\theta_0^*.$$

So

$$\frac{\theta_1^*}{\theta_0^*} = \frac{\theta}{k} = \frac{p_1(\theta)}{k}.$$

$i \geq 2$: Set $j = i - 1$ above. We have

$$c_{i-2}\theta^*_{i-2} + a_{i-1}\theta^*_{i-1} + b_{i-1}\theta^*_i = \theta\theta^*_{i-1}.$$

So,

$$\frac{\theta^*_i}{\theta^*_0} = \frac{\theta\theta^*_{i-1} - a_{i-1}\theta^*_{i-1} - c_{i-1}\theta^*_{i-2}}{b_{i-1}\theta^*_0} \tag{26.15}$$

$$= \left((\theta - a_{i-1})\frac{\theta^*_{i-1}}{\theta^*_0} - c_{i-1}\frac{\theta^*_{i-2}}{\theta^*_0} \right) \frac{1}{b_{i-1}} \tag{26.16}$$

$$= \left((\theta - a_{i-1})\frac{p_{i-1}(\theta)}{kb_1 \cdots b_{i-2}} - c_{i-1}\frac{p_{i-2}(\theta)}{kb_1 \cdots b_{i-3}} \right) \frac{1}{b_{i-1}} \tag{26.17}$$

$$= \frac{p_i(\theta)}{kb_1 \cdots b_{i-2}b_{i-1}}, \tag{26.18}$$

as desired.

$\square$

Chapter 27

P-and Q-Polynomial Schemes

Friday, April 2, 1993

Theorem 27.1. *Let $\Gamma = (X, E)$ be a distance-regular graph of diameter $D \geq 3$.*

Let θ denote an eigenvalue of Γ with associated primitive idempotent

$$E = \frac{1}{|X|} \sum_{i=0}^{D} \theta_i^* A_i.$$

Then the following are equivalent.

(i) Γ is Q-polynomial with respect to E.

(ii) $\theta_0^* \neq \theta_h^*$ for all $h \in \{1, 2, \dots, D\}$ and for $i \in \{3, \dots, D\}$,

$$c_i \left(\theta_2^* - \theta_i^* - \frac{(\theta_1^* - \theta_{i-1}^*)^2}{\theta_0^* - \theta_i^*} \right) + b_{i-1} \left(\theta_2^* - \theta_{i-1}^* - \frac{(\theta_1^* - \theta_i^*)^2}{\theta_0^* - \theta_{i-1}^*} \right) \tag{27.1}$$

$$= (k - \theta)(\theta_1^* + \theta_2^* - \theta_{i-1}^* - \theta_i^*) - (\theta + 1)(\theta_0^* - \theta_2^*) \tag{27.2}$$

(iii) $\theta_0^* \neq \theta_h^*$ for all $h \in \{1, 2, \dots, D\}$ and (27.2) *holds for $i = 3$.*

HS MEMO

Note (27.2) is trivial for $i = 1, 2$.

$i = 1$:

$$\begin{aligned}
\text{LHS} &= \left(\theta_2^* - \theta_1^* - \frac{(\theta_1^* - \theta_0^*)^2}{\theta_0^* - \theta_1^*}\right) + k(\theta_2^* - \theta_0^*) && (27.3)\\
&= \theta_2^* - \theta_1^* - \theta_0^* + \theta_1^* + k(\theta_2^* - \theta_0^*) && (27.4)\\
&= (k+1)(\theta_2^* - \theta_0^*) && (27.5)\\
\text{RHS} &= (k-\theta)(\theta_1^* + \theta_2^* - \theta_0^* - \theta_1^*) - (\theta+1)(\theta_0^* - \theta_2^*) && (27.6)\\
&= (k+1)(\theta_2^* - \theta_0^*). && (27.7)
\end{aligned}$$

$i = 2$:

$$\begin{aligned}
\text{LHS} &= b_1\left(\theta_2^* - \theta_1^* - \frac{(\theta_1^* - \theta_0^*)^2}{\theta_0^* - \theta_1^*}\right) && (27.8)\\
&= b_1\frac{(\theta_2^* - \theta_1^*)(\theta_0^* - \theta_1^* - \theta_2^* + \theta_1^*)}{\theta_0^* - \theta_1^*} && (27.9)\\
&= b_1\frac{(\theta_2^* - \theta_1^*)(\theta_0^* - \theta_2^*)}{\theta_0^* - \theta_1^*} && (27.10)\\
\text{RHS} &= -(\theta+1)(\theta_0^* - \theta_2^*). && (27.11)
\end{aligned}$$

Hence,

$$\begin{aligned}
\text{LHS} = \text{RHS} &\leftrightarrow b_1\frac{\theta_2^* - \theta_1^*}{\theta_0^* - \theta_1^*} + (\theta+1) = 0 && (27.12)\\
&\leftrightarrow b_1(\theta_2^* - \theta_1^*) + (\theta+1)(\theta_0^* - \theta_1^*) = 0. && (27.13)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
b_1\theta_2^* + a_1\theta_1^* + c_1\theta_0^* &= \theta\theta_1^* && (27.14)\\
b_1\theta_1^* + a_1\theta_1^* + c_1\theta_1^* &= k\theta_1^*, && (27.15)
\end{aligned}$$

as $\theta\theta_0^* = k\theta_1^*$. We have

$$b_1(\theta_2^* - \theta_1^*) + (\theta_0^* - \theta_1^*) = \theta(\theta_1^* - \theta_0^*).$$

Proof. Immediate from the proof of Theorem 2.1 in ‘A new inequality for distance-regular graphs’ (Terwilliger, 1995) and Theorem 24.1. □

Note. Suppose $(i) - (iii)$ hold. In particular, $\theta_0^*, \theta_1^*, \dots, \theta_D^*$ are distinct. Then,

$$c_i + a_i + b_i = k \quad (0 \le i \le D).$$

$$c_i\theta_{i-1}^* + a_i\theta_i^* + b_i\theta_{i+1}^* = \theta\theta_j^* \quad (0 \le i \le D).$$

$$\frac{\theta_i^* - \theta_{i+1}^* + \theta_{i+2}^* - \theta_{i-3}^*}{\theta_{i+1}^* - \theta_{i+2}^*} \quad \text{is independent of } i \quad (0 \le i \le D-3).$$

$$c_i\left(\theta_2^*-\theta_i^*-\frac{(\theta_1^*-\theta_{i-1}^*)^2}{\theta_0^*-\theta_i^*}\right)+b_{i-1}\left(\theta_2^*-\theta_{i-1}^*-\frac{(\theta_1^*-\theta_i^*)^2}{\theta_0^*-\theta_{i-1}^*}\right) \tag{27.16}$$
$$=(k-\theta)(\theta_1^*+\theta_2^*-\theta_{i-1}^*-\theta_i^*)-(\theta+1)(\theta_0^*-\theta_2^*). \tag{27.17}$$

Furthermore, we can solve for $c_1,\dots,c_D$, $a_1,\dots,a_D$, $b_0,b_1,\dots,b_{D-1}$ in terms of five free parameters.

In general, we can take the five parameters to be

$$D,q,s^*,r_1,r_2$$

and get

$$b_i=\frac{h(1-q^{i-D})(1-s^*q^{i+1})(1-r_1q^{i+1})(1-r_2q^{i+1})}{(1-s^*q^{2i+1})(1-s^*q^{2i+2})}\quad(0\le i\le D), \tag{27.18}$$
$$c_i=\frac{h(1-q^i)(1-s^*q^{D+i+1})(r_1-s^*q^i)(r_2-s^*q^i)}{s^*q^D(1-s^*q^{2i})(1-s^*q^{2i+1})}\quad(0\le i\le D), \tag{27.19}$$
$$a_i=b_0-c_i-b_i\quad(0\le i\le D), \tag{27.20}$$

where h variable is chosen so that $c_1=1$.

(We must also consider limiting cases $h\to 0$, $s^*\to 0$, $q^*\to\pm 1$.)

See Theorem 2.1 in "The subconstituent algebra of an association scheme, I, II, III, (Terwilliger, 1992), (Terwilliger, 1993a), (Terwilliger, 1993b).

Definition 27.1. Let $\Gamma=(X,E)$ be a distance-regular graph of diameter $D\ge 3$. Choose $q\in\mathbb{R}\ \ \{0,-1\}$, set

$$\begin{bmatrix} i\\ 1\end{bmatrix}=1+q+\dots+q^{i-1}=\begin{cases}\frac{q^i-1}{q-1} & q\neq 1\\ i & q=1.\end{cases}$$

Definition 27.2. Γ has classical parameters if

$$c_i=\begin{bmatrix} i\\ 1\end{bmatrix}\left(1+\alpha\begin{bmatrix} i-1\\ 1\end{bmatrix}\right) \tag{27.21}$$
$$b_i=\left(\begin{bmatrix} D\\ 1\end{bmatrix}-\begin{bmatrix} i\\ 1\end{bmatrix}\right)\left(\sigma-\alpha\begin{bmatrix} i\\ 1\end{bmatrix}\right) \tag{27.22}$$

for some $\sigma,\alpha\in\mathbb{R}$.

(This happens for essentially all known families of distance-regular graphs with unbounded diameter, and is essentially equivalent to $s^*=0$.)

Lemma 27.1. *With above notation, suppose* (27.21), (27.22) *hold. Then,*

(i) $\theta=\frac{b_1}{q}-1$ *is an eigenvalue of* Γ *with* $\theta\neq k$.

(ii) Let $E=|X|^{-1}\sum_{i=0}^{D}\theta_i^*A_i$ *be associated primitive idempotent. Then*

$$\frac{\theta_i^*}{\theta_0^*} = 1 + \left(\frac{\theta}{k} - 1\right) \begin{bmatrix} i \\ 1 \end{bmatrix} q^{1-i} \quad (0 \le i \le D).$$

In particular, $\theta_i^* \neq \theta_0^*$ *for all* $i \in \{1, 2 \dots, D\}$.

(iii) Γ *is* Q*-polynomial with respect to* E.

Proof.

$(i), (ii)$. Need to check

$$c_i\theta_{i-1}^* + a_i\theta_i^* + b_i\theta_{i+1}^* = \theta\theta_i^* \quad (0 \le i \le D),$$

where $a_i = k - c_i - b_i \quad (0 \le i \le D)$.

(equivalently: check

$$c_i(\theta_{i-1}^* - \theta_i^*) + b_i(\theta_i^* - \theta_{i+1}^*) = (\theta - k)\theta_i^* \quad (0 \le i \le D), \tag{27.23}$$

where $c_i, b_i, \theta_i^*, \theta$ are as given.)

HS MEMO

$$\theta = \frac{b_1}{q} - 1,\ \frac{\theta_i^*}{\theta_0^*} = 1 + \left(\frac{\theta}{k} - 1\right) \begin{bmatrix} i \\ 1 \end{bmatrix} q^{1-i},\ b_0 = \begin{bmatrix} D \\ 1 \end{bmatrix} \sigma.$$

$i = 0$.

$$\frac{\theta_i^*}{\theta_0^*} = \frac{\theta}{k}, \quad -k\left(1 - \frac{\theta_1^*}{\theta_0^*}\right) = -k\left(1 - \frac{\theta}{k}\right) = \theta - k.$$

$$\frac{\theta_{i-1}^* - \theta_i^*}{\theta_0^*} = \left(\frac{\theta}{k} - 1\right)\left(\begin{bmatrix} i-1 \\ 1 \end{bmatrix} q^{2-i} - \begin{bmatrix} i \\ 1 \end{bmatrix} q^{1-i}\right) = -\left(\frac{\theta}{k} - 1\right) q^{1-i}.$$

$$\theta - k = \left(\begin{bmatrix} D \\ 1 \end{bmatrix} - 1\right)(\sigma - \alpha)/q - 1 - \begin{bmatrix} D \\ 1 \end{bmatrix} \sigma = \begin{bmatrix} D-1 \\ 1 \end{bmatrix} (\sigma - \alpha) - 1 - \begin{bmatrix} D \\ 1 \end{bmatrix} \sigma.$$

$$
\begin{aligned}
&(c_i(\theta^*_{i-1} - \theta^*_i) + b_i(\theta^*_i - \theta^*_{i+1}) - (\theta - k)\theta^*_i)/\theta^*_0 && (27.24)\\
&\quad = -\begin{bmatrix} i \\ 1 \end{bmatrix}\left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix}\right)\left(\frac{\theta}{k} - 1\right) q^{1-i} + \left(\begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix}\right)\left(\sigma - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}\right)\left(\frac{\theta}{k} - 1\right) q^{-i} && (27.25)\\
&\qquad\quad - (\theta - k)\left(1 + \left(\frac{\theta}{k} - 1\right)\begin{bmatrix} i \\ 1 \end{bmatrix} q^{1-i}\right) && (27.26)\\
&\quad = \left(\frac{\theta}{k} - 1\right)\left(-\begin{bmatrix} i \\ 1 \end{bmatrix}\left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix}\right) q^{1-i} + \begin{bmatrix} D-i \\ 1 \end{bmatrix}\left(\sigma - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}\right)\right. && (27.27)\\
&\qquad\quad \left. - \left(\begin{bmatrix} D \\ 1 \end{bmatrix}\sigma + \left(\begin{bmatrix} D-1 \\ 1 \end{bmatrix}(\sigma - \alpha) - 1 - \begin{bmatrix} D \\ 1 \end{bmatrix}\sigma\right)\begin{bmatrix} i \\ 1 \end{bmatrix} q^{1-i}\right)\right) && (27.28)\\
&\quad = \left(\frac{\theta}{k} - 1\right)\left(-\begin{bmatrix} i \\ 1 \end{bmatrix} q^{1-i} - \alpha\left(\begin{bmatrix} i \\ 1 \end{bmatrix}\begin{bmatrix} i-1 \\ 1 \end{bmatrix} q^{1-i} + \begin{bmatrix} D-i \\ 1 \end{bmatrix}\begin{bmatrix} i \\ 1 \end{bmatrix} - \begin{bmatrix} D-1 \\ 1 \end{bmatrix}\begin{bmatrix} i \\ 1 \end{bmatrix} q^{1-i}\right)\right. && (27.29)\\
&\qquad\quad \left. +\sigma\left(\begin{bmatrix} D-i \\ 1 \end{bmatrix} - \begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} D-1 \\ 1 \end{bmatrix}\begin{bmatrix} i \\ 1 \end{bmatrix} q^{1-i} + \begin{bmatrix} D \\ 1 \end{bmatrix}\begin{bmatrix} i \\ 1 \end{bmatrix} q^{1-i}\right) + \begin{bmatrix} i \\ 1 \end{bmatrix} q^{1-i}\right) && (27.30)
\end{aligned}
$$

Check $\theta \neq k$. Suppose $\theta = k$. Then

$$\frac{b_1}{q} - 1 = k, \quad \text{and} \quad q > 0.$$

By (27.21), (27.22),

$$
\begin{aligned}
qc_i - b_i - q(qc_{i-1} - b_{i-1}) &= (k - \theta)q \quad (1 \leq i \leq D) && (27.31)\\
&= 0. && (27.32)
\end{aligned}
$$

HS MEMO

With the notation of Lemma 27.1, we have the above equality in general.

$$
\begin{aligned}
& qc_i - b_i - q(qc_{i-1} - b_{i-1}) && (27.33)\\
&= q\begin{bmatrix} i \\ 1 \end{bmatrix}\left(1 + \alpha\begin{bmatrix} i-1 \\ 1 \end{bmatrix}\right) - \left(\begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix}\right)\left(\sigma - \alpha\begin{bmatrix} i \\ 1 \end{bmatrix}\right) && (27.34)\\
&\qquad - q\left(q\begin{bmatrix} i-1 \\ 1 \end{bmatrix}\left(1 + \alpha\begin{bmatrix} i-2 \\ 1 \end{bmatrix}\right) - \left(\begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} i-1 \\ 1 \end{bmatrix}\right)\left(\sigma - \alpha\begin{bmatrix} i-1 \\ 1 \end{bmatrix}\right)\right) && (27.35)\\
&= \left(q\begin{bmatrix} i \\ 1 \end{bmatrix} - q^2\begin{bmatrix} i-1 \\ 1 \end{bmatrix}\right) && (27.36)\\
&\qquad + \alpha\left(q\begin{bmatrix} i \\ 1 \end{bmatrix}\begin{bmatrix} i-1 \\ 1 \end{bmatrix} + \begin{bmatrix} D \\ 1 \end{bmatrix}\begin{bmatrix} i \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix}\begin{bmatrix} i \\ 1 \end{bmatrix}\right. && (27.37)\\
&\qquad \left. - q^2\begin{bmatrix} i-1 \\ 1 \end{bmatrix}\begin{bmatrix} i-2 \\ 1 \end{bmatrix} - q\begin{bmatrix} D \\ 1 \end{bmatrix}\begin{bmatrix} i-1 \\ 1 \end{bmatrix} + q\begin{bmatrix} i-1 \\ 1 \end{bmatrix}\begin{bmatrix} i-1 \\ 1 \end{bmatrix}\right) && (27.38)\\
&\qquad + \sigma\left(-\begin{bmatrix} D \\ 1 \end{bmatrix} + \begin{bmatrix} i \\ 1 \end{bmatrix} + q\begin{bmatrix} i-1 \\ 1 \end{bmatrix}\right) && (27.39)\\
&= q + \alpha\left(-\begin{bmatrix} i \\ 1 \end{bmatrix} + \begin{bmatrix} D \\ 1 \end{bmatrix} + q\begin{bmatrix} i-1 \\ 1 \end{bmatrix}\right) + \sigma(q^D - 1 + 1) && (27.40)\\
&= q\left(1 + \begin{bmatrix} D-1 \\ 1 \end{bmatrix}\alpha + q^{D-1}\sigma\right) && (27.41)\\
&= q\left(\begin{bmatrix} D \\ 1 \end{bmatrix}\sigma - \begin{bmatrix} D-1 \\ 1 \end{bmatrix}\sigma + \begin{bmatrix} D-1 \\ 1 \end{bmatrix}\alpha + 1\right) && (27.42)\\
&= q\left(k - \frac{\begin{bmatrix} D \\ 1 \end{bmatrix} - 1}{q}(\sigma - \alpha) + 1\right) && (27.43)\\
&= q(k - \theta). && (27.44)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
qc_i - b_i &= q(qc_{i-1} - b_{i-1}) \quad (1 \le i \le D) && (27.45)\\
&= q^i(qc_0 - b_0) && (27.46)\\
&= -q^i k. && (27.47)
\end{aligned}
$$

If $i = D$, $qc_D = -q^D k$, $c_D = -q^{D-1}k < 0$, a contradiction.

(iii) Check the equation (ii) of Theorem 27.1 holds for $i = 3$.

□

HS MEMO

$\theta_0^* \neq \theta_h^*$ for all $h \in \{1, 2, \ldots, D\}$ and

$$c_3\left(\theta_2^* - \theta_3^* - \frac{(\theta_1^* - \theta_2^*)^2}{\theta_0^* - \theta_3^*}\right) - b_2\frac{(\theta_1^* - \theta_3^*)^2}{\theta_0^* - \theta_2^*} = (k-\theta)(\theta_1^* - \theta_3^*) - (\theta+1)(\theta_0^* - \theta_2^*).$$

Pf.

$$\frac{\text{LHS}}{\theta_0^*} = \begin{bmatrix}3\\1\end{bmatrix}\left(1+\alpha\begin{bmatrix}2\\1\end{bmatrix}\right)\left(1-\frac{\theta}{k}\right)\left(q^{-2} - \frac{q^{-2}}{\begin{bmatrix}3\\1\end{bmatrix}q^{-2}}\right) \tag{27.48}$$

$$-\left(\begin{bmatrix}D\\1\end{bmatrix} - \begin{bmatrix}2\\1\end{bmatrix}\right)\left(\sigma - \alpha\begin{bmatrix}2\\1\end{bmatrix}\right)\left(1-\frac{\theta}{k}\right)\frac{\left(\begin{bmatrix}3\\1\end{bmatrix}q^{1-3} - 1\right)^2}{\begin{bmatrix}2\\1\end{bmatrix}q^{-1}} \tag{27.49}$$

$$= \left(1-\frac{\theta}{k}\right)\left(\left(1+\alpha\begin{bmatrix}2\\1\end{bmatrix}\right)q^{-2}\begin{bmatrix}2\\1\end{bmatrix} - \left(\begin{bmatrix}D\\1\end{bmatrix} - \begin{bmatrix}2\\1\end{bmatrix}\right)\left(\sigma - \alpha\begin{bmatrix}2\\1\end{bmatrix}\right)\begin{bmatrix}2\\1\end{bmatrix}q^{-3}\right) \tag{27.50}$$

$$= \left(1-\frac{\theta}{k}\right)\left(q^{-2}\begin{bmatrix}2\\1\end{bmatrix} + \alpha\left(q^{-2}\begin{bmatrix}2\\1\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} + q^{-1}\begin{bmatrix}2\\1\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}\begin{bmatrix}D-2\\1\end{bmatrix}\right)\right. \tag{27.51}$$

$$\left. -q^{-1}\begin{bmatrix}2\\1\end{bmatrix}\begin{bmatrix}D-2\\1\end{bmatrix}\sigma\right) \tag{27.52}$$

$$\frac{\text{RHS}}{\theta_0^*} = \left(\begin{bmatrix}D\\1\end{bmatrix}\sigma - \begin{bmatrix}D-1\\1\end{bmatrix}(\sigma-\alpha) + 1\right)\left(1-\frac{\theta}{k}\right)\left(\begin{bmatrix}3\\1\end{bmatrix}q^{-2} - 1\right) \tag{27.53}$$

$$-\begin{bmatrix}D-1\\1\end{bmatrix}(\sigma-\alpha)\left(1-\frac{\theta}{k}\right)\begin{bmatrix}2\\1\end{bmatrix}q^{-1} \tag{27.54}$$

$$= \left(1-\frac{\theta}{k}\right)\left(q^{-2}\begin{bmatrix}2\\1\end{bmatrix} + \begin{bmatrix}2\\1\end{bmatrix}q^{-1}\sigma\left(q^{D-2} - \begin{bmatrix}D-1\\1\end{bmatrix}\right)\right. \tag{27.55}$$

$$\left. + \begin{bmatrix}2\\1\end{bmatrix}q^{-2}\alpha\left(\begin{bmatrix}D-1\\1\end{bmatrix} + q\begin{bmatrix}D-1\\1\end{bmatrix}\right)\right) \tag{27.56}$$

$$= \left(1-\frac{\theta}{k}\right)\left(q^{-2}\begin{bmatrix}2\\1\end{bmatrix} - \sigma q^{-1}\begin{bmatrix}2\\1\end{bmatrix}\begin{bmatrix}D-2\\1\end{bmatrix} + \alpha q^{-2}\begin{bmatrix}2\\1\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}\begin{bmatrix}D-1\\1\end{bmatrix}\right) \tag{27.57}$$

Example 27.1. Q-polynomial distance-regular graphs with classical parameters.

D-cube: $c_i = i$, $b_i = D - i$

has classical parameters: $(q, \alpha, \sigma) = (1, 0, 1)$.

Johnson graph $J(D, N)$ $(N \geq 2D)$:

$c_i = i^2$, $b_i = (D-i)(N-D-i)$ has classical parameters $(q, \alpha, \sigma) = (1, 1, N-D)$.

q-analogue of Johnson graph $J_q(D,N)$ $(D \geq 2D)$:

$$c_i = \left(\frac{q^i-1}{q-1}\right)^2 = \begin{bmatrix} i \\ 1 \end{bmatrix}^2, \quad b_i = \frac{q(q^D-q^i)(q^{N-D}-q^i)}{(q-1)^2}$$

has classical parameters

$$(q,\alpha,\sigma) = \left(q, q, \left(\frac{q^{N-D+1}-1}{q-1}\right) - 1\right) = \left(q, q, \begin{bmatrix} N-D+1 \\ 1 \end{bmatrix} - 1\right).$$

HS MEMO

$$\begin{aligned} b_i &= \left(\begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix}\right)\left(\begin{bmatrix} N-D+1 \\ 1 \end{bmatrix} - 1 - q\begin{bmatrix} i \\ 1 \end{bmatrix}\right) &(27.58)\\ &= \left(\begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix}\right)\left(\begin{bmatrix} N-D+1 \\ 1 \end{bmatrix} - \begin{bmatrix} i+1 \\ 1 \end{bmatrix}\right) &(27.59)\\ &= \frac{q(q^D-q^i)(q^{N-D}-q^i)}{(q-1)^2}. &(27.60) \end{aligned}$$

Chapter 28

The First Eigenspace of a Q-DRG

Monday, April 5, 1993

Lemma 28.1. *Let $\Gamma = (X, E)$ be distance-regular of diameter $D \geq 3$ with standard module V. Suppose Γ is Q-polynomial with respect to a primitive idempotent E_1. Pick a vertex $x \in X$. Then*

$$E_1V = \mathrm{Span}\{E_1\hat{y} \mid \partial(x, y) \leq 2\}.$$

In particular,

$$\dim E_1V \leq 1 + k_1 + k_2.$$

Proof. Let $\Delta = \{E_1\hat{y} \mid \partial(x, y) \leq 2\}$.

$E_1V \supseteq \mathrm{Span}\Delta$: clear.

$E_1V \subseteq \mathrm{Span}\Delta$: Pick a vertex $y \in X$. Show that $E_1\hat{y} \in \mathrm{Span}\Delta$.

Induction on $h = \partial(x, y)$.

Case $h \leq 2$.

$E_1\hat{y} \in \mathrm{Span}\Delta$ follows from construction.

Case $h \geq 3$.

Pick a vertex $x' \in X$ such that

$$\partial(x, x') = h - 3, \quad \partial(x', y) = 3.$$

By Theorem 24.1.

$$\sum_{z \in X, (x,z) \in R_1, (y,z) \in R_2} E_1\hat{z} - \sum_{z' \in X, (x,z') \in R_2, (y,z') \in R_1} E_1\hat{z'} = r^3_{12}(E_1\hat{x'} - E_1\hat{y}),$$

$$r^3_{12} = \frac{c_3(\theta^*_1 - \theta^*_2)}{\theta^*_0 - \theta^*_3} \neq 0.$$

So, $E_1\hat{y} \in \text{Span}\{f, g, E_1\hat{x'}\}$, where

$$f = \sum_{z \in X, (x,z) \in R_1, (y,z) \in R_2} E_1\hat{z}, \quad g = \sum_{z' \in X, (x,z') \in R_2, (y,z') \in R_1} E_1\hat{z'}.$$

Observe that each z in the f-sum satisfies $\partial(x, z) = h - 2$.

So, by induction hypothesis

$$E_1\hat{z} \in \text{Span}\Delta, \quad \text{or} \quad f \in \text{Span}\Delta.$$

Observe that each z' in the g-sum satisfies $\partial(x, z') = h - 1$.

So by induction hypothesis

$$E_1\hat{z'} \in \text{Span}\Delta, \quad \text{or} \quad g \in \text{Span}\Delta.$$

Also $\partial(x, x') = h - 3$ implies $E_1\hat{x'} \in \text{Span}\Delta$.

Therefore $E_1\hat{y} \in \text{Span}\Delta$. □

Note. Let Γ, E_1, x be as in Lemma 28.1.

Assume $D \geq 4$.

Observe that there are many linear dependences among

$$\{E\hat{y} \mid y \in \Delta\},$$

where $\Delta = \{y \in X \mid \partial(x, y) \leq 2\}$.

Take any $y \in X$ such that $\partial(x, y) \geq 4$.

More than one choice for x' in the proof of Lemma 28.1 implies

"more than one way to put $E_1\hat{y} \in \text{Span}E_1\Delta$."

Open Problem:

(i) Give a precise description of the linear dependences among

$$\{E_1\hat{y} \mid y \in \Delta\}.$$

(ii) Find a subset $\Delta' \subseteq \Delta$ such that

$$\{E_1\hat{y} \mid y \in \Delta'\}$$

is a basis for E_1V, (or find some other 'nice' basis for E_1V).

Conjecture 28.1. *Let* Γ*,* E_1*,* x *be as in Lemma 28.1. Set*

$$\widetilde{X} = \{y \in X \mid \partial(x,y) \leq 2\}, \tag{28.1}$$

$$\tilde{\partial} = \text{the restriction of the distance function } \partial \text{ to } \widetilde{X}. \tag{28.2}$$

Then Γ *is determined by* $\widetilde{X}$ *and* $\tilde{\partial}$*.*

(There should be some canonical way to reconstruct Γ *from* $\widetilde{X}$ *and* $\tilde{\partial}$*.)*

Chapter 29

Tridiagonal Pair A, A^*

Wednesday, April 7, 1993

Introduction to Theorem 29.1

Let $\Gamma = (X, E)$ be distance-regular with diameter $D \geq 3$.

Assume Γ is Q-polynomial with respect to E_1.

Fix a vertex $x \in X$. Write $E_i^* \equiv E_i^*(x)$, $A_i^* \equiv A_i^*(x)$, $A^* = A_1^*$.

We know for h, i, j $(0 \leq h, i, j \leq D)$,

$$E_i^* A_h E_j^* = O \leftrightarrow p_{ij}^h = 0 \tag{29.1}$$

$$E_i A_h^* E_j = O \leftrightarrow q_{ij}^h = 0. \tag{29.2}$$

Also, for h, i, j $(0 \leq h, i, j \leq D)$,

$$h < |i-j| \rightarrow p_{ij}^h = 0,\ q_{ij}^h = 0 \tag{29.3}$$

$$h = |i-j| \rightarrow p_{ij}^h \neq 0,\ q_{ij}^h \neq 0. \tag{29.4}$$

Some A_h (resp. A_h^*) is a polynomial of degree exactly h in A (resp. A^*), it follows, for h, i, j $(0 \leq h, i, j \leq D)$,

$$E_i^* A^h E_j^*,\ E_i {A^*}^h E_j \quad \begin{cases} = 0 & \text{if } h < |i-j|, \\ \neq 0 & \text{if } h = |i-j|. \end{cases}$$

We saw that there exist $\beta, \gamma, \delta \in \mathbb{R}$ such that

$$0 = [A, A^2 A^* - \beta A A^* A + A^* A^2 - \gamma(A A^* + A^* A) - \delta A^*].$$

In fact, there exist $\beta, \gamma^*, \delta^* \in \mathbb{R}$ such that

$$0 = [A^*, {A^*}^2 A - \beta A^* A A^* + A {A^*}^2 - \gamma^*(A^* A + A A^*) - \delta^* A]$$

as well as we will now show.

Let K denote any field. Let V denote any vector space over K of finite positive dimension. Let $\mathrm{End}_K(V)$ denote the K-algebra of all K-linear transformations $V \to V$.

Theorem 29.1. *Given semi-simple elements $A, A^* \in \mathrm{End}_K(V)$, suppose*

$$E_i (A^*)^h E_j \begin{cases} = 0 & \text{if } h < |i-j|, \\ \neq 0 & \text{if } h = |i-j|. \end{cases} \quad (0 \leq h, i, j \leq D) \tag{29.5}$$

$$E_i^* A^h E_j^* \begin{cases} = 0 & \text{if } h < |i-j|, \\ \neq 0 & \text{if } h = |i-j|. \end{cases} \quad (0 \leq h, i, j \leq R) \tag{29.6}$$

for some ordering $E_0, E_1, \ldots, E_D$ of the primitive idempotents for A, and some ordering $E_0^*, E_1^*, \ldots, E_R^*$ of primitive idempotents for A^*. Then

(i) $R = D$.

(ii) There exist $\beta, \gamma, \gamma^*, \delta, \delta^* \in \mathbb{K}$ such that

$$\begin{aligned} 0 &= [A, A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \delta A^*] && (29.7) \\ &= A^3A^* - A^*A^3 - (\beta+1)(A^2A^*A - AA^*A^2) && (29.8) \\ &\quad - \gamma(A^2A^* - A^*A^2) - \delta(AA^* - A^*A) && (29.9) \end{aligned}$$

$$\begin{aligned} 0 &= [A^*, {A^*}^2A - \beta A^*AA^* + A{A^*}^2 - \gamma^*(A^*A + AA^*) - \delta^* A] && (29.10) \\ &= {A^*}^3A - A{A^*}^3 - (\beta+1)({A^*}^2AA^* - A^*A{A^*}^2) && (29.11) \\ &\quad - \gamma^*({A^*}^2A - A{A^*}^2) - \delta^*(A^*A - AA^*). && (29.12) \end{aligned}$$

(iii) Let θ_i (resp. θ_i^*) denote the eigenvalue of A (resp. A^*) associated with E_i (resp. E_i^*). Then,

$$\begin{aligned} \beta &= \frac{\theta_i - \theta_{i+1} + \theta_{i+2} - \theta_{i+3}}{\theta_{i+1} - \theta_{i+2}} \quad (0 \leq i \leq D-3) && (29.13) \\ &= \frac{\theta_i^* - \theta_{i+1}^* + \theta_{i+2}^* - \theta_{i+3}^*}{\theta_{i+1}^* - \theta_{i+2}^*} \quad (0 \leq i \leq D-3) && (29.14) \\ \gamma &= \theta_i - \beta\theta_{i+1} + \theta_{i+2} \quad (0 \leq i \leq D-2) && (29.15) \\ \gamma^* &= \theta_i^* - \beta\theta_{i+1}^* + \theta_{i+2}^* \quad (0 \leq i \leq D-2) && (29.16) \\ \delta &= \theta_i^2 - \beta\theta_i\theta_{i+1} + \theta_{i+1}^2 - \gamma(\theta_i + \theta_{i+1}) \quad (0 \leq i \leq D-1) && (29.17) \\ \delta^* &= {\theta^*}_i^2 - \beta{\theta^*}_i{\theta^*}_{i+1} + {\theta^*}_{i+1}^2 - \gamma^*({\theta^*}_i + {\theta^*}_{i+1}) \quad (0 \leq i \leq D-1) && (29.18) \end{aligned}$$

In particular, $\beta, \gamma, \gamma^*, \delta, \delta^*$ are uniquely determined by A, A^* and the above ordering of their primitive idempotents, whenever $D \geq 3$.

Proof.

(i) By symmetry, it suffices to show $D \geq R$. Suppose $R > D$.

Since A is semisimple with exactly $D+1$ distinct eigenvalues, the minimal polynomial of A has degree $D+1$.

Since $R \geq D+1$,
$$A^R \in \operatorname{Span}\{A^j \mid 0 \leq j \leq D\}.$$
Multiplying each term on the left by E^*_R and on the right by E^*_0, we find

$$E^*_R A^R E^*_0 \in \operatorname{Span}\{E^*_R A^j E^*_0 \mid 0 \leq j \leq D\}. \tag{29.19}$$

But by (29.6), the left side of (29.19) is nonzero and the right side of (29.19) is 0, a contradiction.

Hence $D \geq R$.

$(ii), (iii)$

Recalling the definitions, we have

$$A = \sum_{i=0}^{D} \theta_i E_i, \tag{29.20}$$
$$A^* = \sum_{i=0}^{D} \theta^*_i E^*_i, \tag{29.21}$$
$$AE_i = E_i A = \theta_i E_i \quad (0 \leq i \leq D), \tag{29.22}$$
$$A^* E^*_i = E^*_i A^* = \theta^*_i E^*_i \quad (0 \leq i \leq D). \tag{29.23}$$

Claim 1. For all integers i, j, k, ℓ $(0 \leq i, j, k, \ell \leq D)$ such that $j + k \leq i - \ell$,

$$E^*_i A^j A^* A^k E^*_\ell = \begin{cases} \theta^*_{\ell+k} E^*_i A^{j+k} E^*_\ell & \text{if } j+k = i-l, \\ O & \text{if } j+k < i-\ell. \end{cases} \tag{29.24}$$

Proof of Claim 1. The product (29.24) eqia;s

$$E^*_i A^j \left(\sum_{h=0}^{D} \theta^*_h E^*_h \right) A^k E^*_\ell = \sum_{h=0}^{D} \theta^*_h E^*_i A^j E^*_h A^k E^*_\ell.$$

Now pick any h $(0 \leq h \leq D)$, where

$$E^*_i A^j E^*_h A^k E^*_\ell \neq O.$$

Then by (29.6), $j \geq |i-h|$, otherwise

$$E^*_i A^j E^*_h = O$$

and by (29.5), $k \geq |h-\ell|$ otherwise

$$E^*_h A^k E^*_\ell = O.$$

Hence,

$$j+k \geq |i-h| + |h-\ell| \geq |i-\ell| \geq i-\ell.$$

Now if $j+k < i-\ell$, we see there is no such h, so (29.24) holds.

(*Pf.* Suppose $i = j+k+\ell$ with $0 \leq i, j, k, \ell, h \leq D$.

Then $i \geq j, k, \ell$. Since $k = |h-\ell|$, if $h \neq \ell+k$, $h = \ell-k$ and $j-i-h$, $\ell-h+i-h = i-\ell$ implies $h=\ell$, $k=0$ and $h=\ell+k$.)

This proves Claim 1.

Let M denote the subalgebra of $\mathrm{End}_K(V)$ generated by A. Observe that M has a basis $E_0, \dots, E_D$ as a vector space over K. Set

$$L := \mathrm{Span}\{mA^*m - nA^*m \mid m, n \in M\}.$$

Claim 2. $\dim L \leq D$.

Proof of Claim 2. Since $E_0, \dots, E_D$ span M,

$$\begin{aligned} L &= \mathrm{Span}\{E_i A^* E_j - E_j A^* E_i \mid 0 \leq i < j \leq D\} && (29.25)\\ &= \mathrm{Span}\{E_{j-1} A^* E_j - E_j A^* E_{j-1} \mid 1 \leq j \leq D\} && (29.26)\end{aligned}$$

by (29.5).

In particular, L has a spanning set of order D.

So, Claim 2 holds.

Claim 3. $\{A^i A^* - A^* A^i \mid 1 \leq i \leq D\}$ is a basis for L.

Proof of Claim 3. Since

$$A^i A^* - A^* A^i = A^i A^* I - I A^* A^i$$

is contained in L ($1 \leq i \leq D$), and since $\dim L \leq D$, it suffices to show the given elements are linearly independent.

Suppose they are dependent. Then there exists an integer i ($1 \leq i \leq D$) such that

$$A^i A^* - A^* A^i \in \mathrm{Span}(A^j A^* - A^* A^j \mid 1 \leq j < i). \tag{29.27}$$

Multiplying each term in (29.27) on the left by E^*_i, and on the left by E^*_0, and simplifying using

$$E^*_i (A^\ell A^* - A^* A^\ell) E^*_0 = (\theta^*_0 - \theta^*_i) E^*_i A^\ell E^*_0,$$

we find

$$E^*_i A^\ell E^*_0 \in \mathrm{Span}(E^*_i A^j E^*_0 \mid 1 \leq j < i). \tag{29.28}$$

But the left side of (29.28) is nonzero.

A contradiction.

Since $A^2A^*A - AA^*A^2$ is contained in L, we find by Claim 2,

$$A^2A^*A - AA^*A^2 = \sum_{i=1}^{D} \alpha_i (A^iA^* - A^*A^i) \tag{29.29}$$

for some $\alpha_0, \ldots, \alpha_D \in K$.

Claim 4. $\alpha_i = 0 \quad (3 < i \leq D)$.

Proof of Claim 4. Suppose not, and set

$$t = \max\{i \mid 3 < i \leq D,\ \alpha_i \neq 0\}.$$

Then by (29.29), and Claim 1,

$$0 = E_t^* \left(A^2A^*A - AA^*A^2 - \sum_{i=1}^{D} \alpha_i (A^iA^* - A^*A^i) \right) E_0^* \tag{29.30}$$
$$= \alpha_t(\theta_t^* - \theta_0^*)E_t^*A^tE_0^* \tag{29.31}$$
$$\neq O. \tag{29.32}$$

(Since $\alpha_i = 0$ if $i > t$,

$$E_t^*A^2A^*AE_0^* = E_t^*AA^*A^2E_0^* = O \quad (\text{as } 2+1 < t-0) \tag{29.33}$$
$$E_t^*A^iA^*E_0^* = E_t^*A^*A^iE_0^* = O \tag{29.34}$$
$$E_t^*A^tA^*E_0^* = \theta_0^*E_t^*A^tE_0^*, \tag{29.35}$$
$$E_t^*A^*A^tE_0^* = \theta_t^*E_t^*A^*A^tE_0^*.) \tag{29.36}$$
$$\tag{29.37}$$

A contradiction. This proves Claim 4.

Claim 5. Suppose $D \geq 3$. Then

$$\alpha_3 = \frac{\theta_{i+1}^* - \theta_{i+2}^*}{\theta_i^* - \theta_{i+3}^*} \quad \text{for all } i,\ (0 \leq i \leq D-3). \tag{29.38}$$

In particular, $\alpha \neq 0$.

Proof of Claim 5. Fix and integer i $(0 \leq i \leq D-3)$. Then by (29.24) and (29.29),

$$O = E_{i+3}^* \left(A^2A^*A - AA^*A^2 - \sum_{j=1}^{3} \alpha_j (A^iA^* - A^*A^i) \right) E_i^* \tag{29.39}$$
$$= (\theta_{i+1}^* - \theta_{i+2}^* - \alpha_3(\theta_i^* - \theta_{i+3}^*))E_{i+3}^*A^3E_i^*. \tag{29.40}$$

But $E^*_{i+3}A^3E^*_i \neq O$ by (29.6), so (29.38) holds.

This proves Claim 5.

Claim 6. Lines (29.7), (29.9), (29.14) hold.

Proof of Claim 6. First suppose $D \geq 3$. Then by (29.29), Claims 4, and 5,

$$A^2A^*A - AA^*A^2 = \alpha_3(A^3A^* - A^*A^3) + \alpha_2(A^2A^* - A^*A^2) + \alpha_1(AA^* - A^*A), \tag{29.41}$$

where $\alpha_3 \neq 0$. Hence

$$A^3A^* - A^*A^3 - \frac{1}{\alpha_3}(A^2A^*A - AA^*A^2) + \frac{\alpha_2}{\alpha_3}(A^2A^* - A^*A^2) + \frac{\alpha_1}{\alpha_3}(AA^* - A^*A) = O.$$

Now (29.9) is immediate, where

$$\beta = \frac{1}{\alpha_3} - 1, \tag{29.42}$$

$$\gamma = -\frac{\alpha_2}{\alpha_3}, \tag{29.43}$$

$$\delta = -\frac{\alpha_1}{\alpha_3}. \tag{29.44}$$

The line (29.7) follows from the definition of [,].

The line (29.14) is immediate from (29.38) and (29.42).

Now suppose $D < 3$. Then the line (29.14) is vacuously true, so consider (29.9).

Let α_3 denote any nonzoro element of K.

Then $A^2A^* - A^*A^2, AA^* - A^*A$ certainly span L by Claim 3.

So, (29.41) holds for appropriate α_1 and $\alpha_2 \in K$.

Now, (29.9) holds, where β, γ, δ are given by (29.42), (29.43), (29.44).

Claim 7. Lines (29.13), (29.15), (29.17) hold.

Proof of Claim 7. Pick an integer i $(0 \leq i \leq D - 1)$.

By (29.9), we have

$$O = E_i(A^3A^* - A^*A^3 - (\beta+1)(A^2A^*A - AA^*A^2) - \gamma(A^2A^* - A^*A^2) - \delta(AA^* - A^*A))E_{i+1} \tag{29.45}$$

$$= E_iA^*E_{i+1}(\theta_i^3 - \theta_{i+1}^3 - (\beta+1)(\theta_i^*\theta_{i+1} - \theta_i\theta_{i+1}^2) - \gamma(\theta_i^2 - \theta_{i+1}^2) - \delta(\theta_i - \theta_{i+1})) \tag{29.46}$$

$$= E_iA^*E_{i+1}(\theta_i - \theta_{i+1})(\theta_i^2 + \theta_i\theta_{i+1} + \theta_{i+1}^2 - (\beta+1)\theta_i\theta_{i+1} - \gamma(\theta_i + \theta_{i+1}) - \delta) \tag{29.47}$$

$$= E_iA^*E_{i+1}(\theta_i - \theta_{i+1})(\theta_i^2 - \beta\theta_i\theta_{i+1} + \theta_{i+1}^2 - \gamma(\theta_i + \theta_{i+1}) - \delta). \tag{29.48}$$

But $E_iA^*E_{i+1} \neq O$ by (29.5), and of course, $\theta_i \neq \theta_{i+1}$, so

$$0 = \theta_i^2 - \beta\theta_i\theta_{i+1} + \theta_{i+1}^2 - \gamma(\theta_i + \theta_{i+1}) - \delta.$$

This proves (29.17).

To obtain (29.15), pick any integer i $(0 \leq i \leq D-2)$. Then by (29.17),

$$\begin{aligned}
0 &= \theta_i^2 - \beta\theta_i\theta_{i+1} + \theta_{i+1}^2 - \gamma(\theta_i + \theta_{i+1}) - \delta && (29.49)\\
&\quad - (\theta_{i+1}^2 - \beta\theta_{i+1}\theta_{i+2} + \theta_{i+2}^2 - \gamma(\theta_{i+1} + \theta_{i+2}) - \delta) && (29.50)\\
&= \theta_i^2 - \beta\theta_i\theta_{i+1} - \gamma\theta_i + \beta\theta_{i+1}\theta_{i+2} - {\theta_{i+2}}^2 + \gamma\theta_{i+2} && (29.51)\\
&= (\theta_i - \theta_{i+2})(\theta_i - \beta\theta_{i+1} + \theta_{i+2} - \gamma). && (29.52)
\end{aligned}$$

So $0 = \theta_i - \beta\theta_{i+1} + \theta_{i+2} - \gamma$.

This gives (29.15).

To see (29.13), pick an integer i $(0 \leq i \leq D-3)$.

Then by (29.15),

$$\begin{aligned}
0 &= (\theta_i - \beta\theta_{i+1} + \theta_{i+2} - \gamma) - (\theta_{i+1} - \beta\theta_{i+2} + \theta_{i+3} - \gamma) && (29.53)\\
&= \theta_i - (\beta+1)\theta_{i+1} + (\beta+1)\theta_{i+2} - \theta_{i+3}. && (29.54)
\end{aligned}$$

We have

$$\beta = \frac{\theta_i - \theta_{i+3}}{\theta_{i+1} - \theta_{i+2}} - 1 = \frac{\theta_i - \theta_{i+1} + \theta_{i+2} - \theta_{i+3}}{\theta_{i+1} - \theta_{i+2}},$$

as desired.

This proves Claim 7.

We have now proved (29.7), (29.9), (29.13), (29.14), (29.15), (29.17).

Interchanging the roles of A and A^*, we obtain (29.10), (29.12), (29.16), (29.18).

□

Chapter 30

R, F, L Matrices

Monday, April 12, 1993

Let $\Gamma = (X, E)$ be distance regular of diameter $D \geq 3$ with standard module V.

Assume Γ is Q-polynomial with respect to the ordering

$$E_0, E_1, \ldots, E_D$$

of primitive idempotents. Let A_i be an i-th adjacency matrix, and $A = A_1$.

$$A = \sum_{i=0}^{D} \theta_i A_i, \quad E_i = |X|^{-1} \sum_{i=0}^{D} \theta_i^* A_i.$$

Fix a vertex $x \in X$, write

$$E_i^* \equiv E_i^*(x), \quad A_i^* \equiv A_i^*(x), \quad A^* \equiv A_1^*, \quad T \equiv T(x).$$

Then

$$A^* = \sum_{i=0}^{D} \theta_i^* E_i^*.$$

By Theorem 29.1, there exist $\beta, \gamma, \gamma^*, \delta, \delta^* \in \mathbb{R}$ such that

$$0 = [A, A^2 A^* - \beta A A^* A + A^* A^2 - \gamma(AA^* + A^* A) - \delta A^*] \tag{30.1}$$

$$0 = [A^*, {A^*}^2 A - \beta^* A^* A A^* + A {A^*}^2 - \gamma^*(A^* A + AA^*) - \delta^* A] \tag{30.2}$$

Recall raising matrix

$$R = \sum_{i=0}^{D} E_{i+1}^* A E_i^*$$

satisfies

$$R(E_i^* V) \subseteq E_{i+1}^* V \quad (0 \leq i \leq D), \quad E_{D+1}^* V = 0,$$

lowering matrix

$$L = \sum_{i=0}^{D} E_{i-1}^* A E_i^*$$

satisfies

$$L(E_i^* V) \subseteq E_{i-1}^* V \quad (0 \leq i \leq D), \quad E_{-1}^* V = 0,$$

and flat matrix

$$F = \sum_{i=0}^{D} E_i^* A E_i^*$$

satisfies

$$F(E_i^* V) \subseteq E_i^* V \quad (0 \leq i \leq D).$$

Also,

$$A = R + F + L.$$

Theorem 30.1. *With the above notation and assumptions,*
(i) For all i $(2 \leq i \leq D)$,

$$g_i^- F L^2 + LFL + g_i^+ L^2 F - \gamma L^2) E_i^* = O,$$

where

$$g_i^+ = \frac{\theta_{i-2}^* - (\beta+1)\theta_{i-1}^* + \beta\theta_i^*}{\theta_{i-2}^* - \theta_i^*} \tag{30.3}$$

$$g_i^- = \frac{\theta_{i-2}^* + (\beta+1)\theta_{i-1}^* - \theta_i^*}{\theta_{i-2}^* - \theta_i^*}. \tag{30.4}$$

(ii) For all i $(0 \leq i \leq D)$,

$$[F, LR - h_i RL] E_i^* = O,$$

where

$$h_i = \frac{\theta_{i-1}^* - \theta_i^*}{\theta_i^* - \theta_{i+1}^*} \quad (1 \leq i \leq D-1), \tag{30.5}$$

and h_0, h_D are indeterminants.
(iii) For all i $(1 \leq i \leq D)$,

$$(e_i^- RL^2 + (\beta+2) LRL + e_i^+ L^2 R + LF^2 - \beta FLF + F^2 L - \gamma(LF + FL) - \delta L) E_i^* = O,$$

where

$$e_i^+ = \frac{\theta^*_{i-1} - (\beta+2)\theta^*_i + (\beta+1)\theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i} \quad (1 \le i \le D) \tag{30.6}$$

$$e_i^- = \frac{-(\beta+1)\theta^*_{i-2} + (\beta+2)\theta^*_{i-1} - \theta^*_i}{\theta^*_{i-1} - \theta^*_i} \quad (2 \le i \le D), \tag{30.7}$$

and e_0^+, e_1^- *are indeterminants.*

Proof. We have

$$O = A^3A^* - A^*A^3 - (\beta+1)(A^2A^*A - AA^*A^2) - \gamma(A^2A^* - A^*A^2) - \delta(AA^* - A^*A).$$

(i) Fix i $(2 \le i \le D)$, and multply above on the left by E^*_{i-2}, and on the right by E^*_i. Now reduce.

For example,

$$E^*_{i-2}A^3A^*E^*_i = \theta^*_i E^*_{i-2}A^3E^*_i,$$

where

$$\begin{aligned}
E^*_{i-2}A^3E^*_i &= E^*_{i-2}A\left(\sum_{r=0}^{D} E^*_r\right)A\left(\sum_{s=0}^{D} E^*_s\right)AE^*_i && (30.8)\\
&= \sum_{r,s} E^*_{i-2}AE^*_rAE^*_sAE^*_i && (30.9)\\
&= \sum_{r,s,|i-2-r|\le 1,|r-s|\le 1,|s-i|\le 1} E^*_{i-2}AE^*_rAE^*_sAE^*_i && (30.10)\\
&= E^*_{i-2}AE^*_{i-2}AE^*_{i-1}AE^*_i + E^*_{i-2}AE^*_{i-1}AE^*_{i-1}AE^*_i + E^*_{i-2}AE^*_{i-1}AE^*_iAE^*_i && (30.11)\\
&= (FL^2 + LFL + L^2F)E^*_i. && (30.12)
\end{aligned}$$

Reducing the other terms in a similar manner, and simplifying, we obtain (i).

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$$\begin{aligned}
E^*_{i-2}A^*A^3E^*_i &= \theta^*_{i-2}E^*_{i-2}A^3E^*_i && (30.13)\\
&= \theta^*_{i-2}(FL^2 + LFL + L^2F)E^*_i, && (30.14)\\
E^*_{i-2}A^2A^*AE^*_i &= (\theta^*_{i-1}(FL^2 + LFL) + \theta^*_i L^2F)E^*_i && (30.15)\\
E^*_{i-2}AA^*A^2E^*_i &= (\theta^*_{i-2}FL^2 + \theta^*_{i-1}(LFL + L^2F))E^*_i, && (30.16)\\
E^*_{i-2}(A^2A^* - A^*A^2)E^*_i &= (\theta^*_i - \theta^*_{i-2})L^2E^*_i, && (30.17)\\
E^*_{i-2}(AA^* - A^*A)E^*_i &= O. && (30.18)
\end{aligned}$$

Then we have

$$
\begin{aligned}
O &= ((\theta^*_i - \theta^*_{i-2})(FL^2 + LFL + L^2F) && (30.19)\\
&\quad - (\beta+1)(\theta^*_{i-1}(FL^2 + LFL) + \theta^*_i L^2F - \theta^*_{i-2}FL^2 - \theta^*_{i-1}(LFL + L^2F)) && (30.20)\\
&\quad - \gamma(\theta^*_i - \theta^*_{i-2})L^2)E^*_i && (30.21)\\
&= ((\theta^*_i - \theta^*_{i-2} - (\beta+1)(\theta^*_{i-1} - \theta^*_{i-2}))FL^2 + (\theta^*_i - \theta^*_{i-2})LFL && (30.22)\\
&\quad + (\theta^*_i - \theta^*_{i-2} - (\beta+1)(\theta^*_i - \theta^*_{i-1}))L^2F - \gamma(\theta^*_i - \theta^*_{i-2})L^2)E^*_1 && (30.23)\\
&= -(\theta^*_{i-2} - \theta^*_i)\left(\left(\frac{-\beta\theta^*_{i-2} + (\beta+1)\theta^*_{i-1} - \theta^*_i}{\theta^*_{i-2} - \theta^*_i}\right)FL^2 + LFL\right. && (30.24)\\
&\quad \left. + \left(\frac{\theta^*_{i-2} - (\beta+1)\theta^*_{i-1} + \beta\theta^*_i}{\theta^*_{i-2} - \theta^*_i}\right)L^2F - \gamma L^2\right)E^*_i && (30.25)\\
&= (\theta^*_i - \theta^*_{i-2})(g^-_i FL^2 + LFL + g^+_i L^2F - \gamma L^2)E^*_i. && (30.26)
\end{aligned}
$$

(ii), (iii) are obtained in a similar manner replacing $i-2$ by i (resp. $i-1$).

□

HS MEMO

(ii) We have

$$O = E^*_i(A^3A^* - A^*A^3 - (\beta+1)(A^2A^*A - AA^*A^2) - \gamma(A^2A^* - A^*A^2) - \delta(AA^* - A^*E))E^*_i.$$

Since $\beta + 1 \neq 0$, by (29.42) if $D \geq 3$,

$$
\begin{aligned}
O &= E^*_i(A^2A^*A - AA^*A^2)E^*_i && (30.27)\\
&= ((\theta^*_i - \theta^*_{i-1})RLF + (\theta^*_i - \theta^*_{i+1})LRF) + (\theta^*_{i-1} - \theta^*_i)FRL + (\theta^*_{i+1} - \theta^*_i)FLR)E^*_i && (30.28)\\
&= [F, (\theta^*_{i-1} - \theta^*_i)RL - (\theta^*_i - \theta^*_{i+1})LR]E^*_i && (30.29)\\
&= (\theta^*_{i+1} - \theta^*_i)\left[F, LR - \frac{\theta^*_{i-1} - \theta^*_i}{\theta^*_i - \theta^*_{i+1}}RL\right]E^*_i && (30.30)\\
&= (\theta^*_{i+1} - \theta^*_i)[F, LR - h_iRL]E^*_i. && (30.31)
\end{aligned}
$$

(iii) We have

$$
\begin{aligned}
O = E^*_{i-1}&(A^3A^* - A^*A^3 - (\beta+1)(A^2A^*A - AA^*A^2) - \gamma(A^2A^* - A^*A^2) - \delta(AA^* - A^*A))E^*_i && (30.32)\\
&= ((\theta^*_i - \theta^*_{i-1})(RL^2 + LRL + L^2R + LF^2 + FLF + F^2L)) && (30.33)\\
&\quad - (\beta+1)((\theta^*_{i-1} - \theta^*_{i-2})RL^2 + (\theta^*_{i-1} - \theta^*_i)LRL + (\theta^*_{i+1} - \theta^*_i)L^2R && (30.34)\\
&\quad + (\theta^*_i - \theta^*_{i-1})FLF && (30.35)\\
&\quad - \gamma(\theta^*_i - \theta^*_{i-1})(LF + FL) && (30.36)\\
&\quad - \delta(\theta^*_i - \theta^*_{i-1})L)E^*_i && (30.37)\\
&= ((\theta^*_i - \theta^*_{i-1}) - (\beta+1)(\theta^*_{i-1} - \theta^*_{i-2}))RL^2 && (30.38)\\
&\quad + ((\theta^*_i - \theta^*_{i-1}) - (\beta+1)(\theta^*_{i-1} - \theta^*_i))LRL && (30.39)\\
&\quad + ((\theta^*_i - \theta^*_{i-1}) - (\beta+1)(\theta^*_{i+1} - \theta^*_i))L^2R && (30.40)\\
&\quad + (\theta^*_i - \theta^*_{i-1})LF^2 + (\theta^*_i - \theta^*_{i-1})F^2L && (30.41)\\
&\quad + (\theta^*_i - \theta^*_{i-1} - (\beta+1)(\theta^*_i - \theta^*_{i-1}))FLF && (30.42)\\
&\quad - \gamma(\theta^*_i - \theta^*_{i-1})(LF + FL) && (30.43)\\
&\quad - \delta(\theta^*_i - \theta^*_{i-1})L)E^*_i && (30.44)\\
&= (\theta^*_i - \theta^*_{i-1})\Bigg(\frac{-(\beta+1)\theta^*_{i-2} + (\beta+2)\theta^*_{i-2} - \theta^*_i}{\theta^*_{i-1} - \theta^*_i}RL^2 + (\beta+2)LRL && (30.45)\\
&\qquad + \frac{\theta^*_{i-1} - (\beta+2)\theta^*_i + (\beta+1)\theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i}L^2R + LF^2 - \beta FLF + F^2L && (30.46)\\
&\qquad - \gamma(LF + FL) - \delta L\Bigg)E^*_i && (30.47)\\
&= (e^-_i RL^2 + (\beta+2)LRL + e^+_i L^2R + LF^2 - \beta FLF + F^2L - \gamma(LF + FL) - \delta L)E^*_i && (30.48)\\
&= O. && (30.49)
\end{aligned}
$$

Lemma 30.1. *With the notation of Theorem 30.1,*

$$
\begin{aligned}
e^+_i &= \frac{\theta^*_i - \theta^*_{i+2}}{\theta^*_i - \theta^*_{i-1}} \quad (1 \le i \le D-2) && (30.50)\\
e^-_i &= \frac{\theta^*_{i-1} - \theta^*_{i-3}}{\theta^*_{i-1} - \theta^*_i} \quad (3 \le i \le D) && (30.51)\\
g^+_i &= \frac{\theta^*_i - \theta^*_{i+1}}{\theta^*_i - \theta^*_{i-2}} \quad (2 \le i \le D-1) && (30.52)\\
g^-_i &= \frac{\theta^*_{i-2} - \theta^*_{i-3}}{\theta^*_{i-2} - \theta^*_i} \quad (3 \le i \le D). && (30.53)
\end{aligned}
$$

In particular, $e^\pm_i$, $g^\pm_i$ *are non-zero for the range of* i *given above.*

Proof. In each case, equate the above expression with the corresponding expression in Theorem 30.1. The resulting equation is equal to (29.13). □

HS MEMO

By Corollary 26.1 and Therem 29.1,

$$e_i^+ = \frac{\theta^*_{i-1} - (\beta+2)\theta^*_i + (\beta+1)\theta^*_{i+1}}{\beta^*_{i-1} - \theta^*_i},$$

and

$$\beta + 1 = \frac{\theta^*_{j-1} - \theta^*_j + \theta^*_{j+1} - \theta^*_{j+2}}{\theta^*_j - \theta^*_{j+1}} + 1 = \frac{\theta^*_{j-1} - \theta^*_{j+2}}{\theta^*_j - \theta^*_{j+1}}.$$

Hence,

$$e_i^+ = \frac{1}{\theta^*_{i-1} - \theta^*_i}(\theta^*_{i-1} - \theta^*_i - (\beta+1)(\theta^*_i - \theta^*_{i+1})) \tag{30.54}$$

$$= \frac{1}{\theta^*_{i-1} - \theta^*_i}(\theta^*_{i-1} - \theta^*_i - (\theta^*_{i-1} - \theta^*_{i+2})) \tag{30.55}$$

$$= \frac{\theta^*_i - \theta^*_{i+2}}{\theta^*_i - \theta^*_{i-1}}, \tag{30.56}$$

$$e_i^- = \frac{1}{\theta^*_{i-1} - \theta^*_i}(-(\beta+1)\theta^*_{i-2} + (\beta+2)\theta^*_{i-1} - \theta^*_i) \tag{30.57}$$

$$= \frac{1}{\theta^*_{i-1} - \theta^*_i}(\theta^*_{i-1} - \theta^*_i - \theta^*_{i-3} + \theta^*_i) \tag{30.58}$$

$$= \frac{\theta^*_{i-1} - \theta^*_{i-3}}{\theta^*_{i-1} - \theta^*_i}, \tag{30.59}$$

$$g_i^+ = \frac{1}{\theta^*_{i-2} - \theta^*_i}(\theta^*_{i-2} - (\beta+1)\theta^*_{i-1} + \beta\theta^*_i) \tag{30.60}$$

$$= \frac{1}{\theta^*_i - \theta^*_{i-2}}(\theta^*_i - \theta^*_{i-2} + \theta^*_{i-2} - \theta^*_{i+1}) \tag{30.61}$$

$$= \frac{\theta^*_i - \theta^*_{i+1}}{\theta^*_i - \theta^*_{i-2}}, \tag{30.62}$$

$$g_i^- = \frac{1}{\theta^*_{i-2} - \theta^*_i}(-\beta\theta^*_{i-2} + (\beta+1)\theta^*_{i-1} - \theta^*_i) \tag{30.63}$$

$$= \frac{1}{\theta^*_{i-2} - \theta^*_i}(\theta^*_{i-2} - \theta^*_i + \theta^*_i - \theta^*_{i-3}) \tag{30.64}$$

$$= \frac{\theta^*_{i-2} - \theta^*_{i-3}}{\theta^*_{i-2} - \theta^*_i}. \tag{30.65}$$

Corollary 30.1. *Let $\Gamma = (X, E)$ be dostance-regular of diameter $D \geq 3$, Q-polynomial with respect to $E_0, E_1, \ldots, E_D$. Fix a vertex $x \in X$, write $E^*_i \equiv E^*_i(x), R \equiv R(x), L \equiv L(x), F \equiv F(x)$. Then the following hold.*

(i) $FR^2E^*_j \in \mathrm{Span}(RFRE^*_j, R^2FE^*_j, R^2E^*_j)$, $(0 \leq j \leq D-3)$.

(ii) $R^2FE^*_j \in \mathrm{Span}(RFRE^*_j, FR^2E^*_j, R^2E^*_j)$, $(1 \leq j \leq D-2)$.

(*iii*) $LR^2E^*_j \in \mathrm{Span}(RLRE^*_j, R^2LE^*_j, F^2RE^*_j, FRFE^*_j, RF^2E^*_j, RFE^*_j, FRE^*_j, RE^*_j)$, $(0 \le j \le D-3)$.

(*iv*) $R^2LE^*_j \in \mathrm{Span}(RLRE^*_j, LR^2E^*_j, F^2RE^*_j, FRFE^*_j, RF^2E^*_j, RFE^*_j, FRE^*_j, RE^*_j)$, $(1 \le j \le D)$.

Proof. Immediate from Theorem 30.1, and Lemma 30.1. □

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By Theorem 30.1, and Lemma 30.1, we have the following, but similarly we can obtain above.

(*i*) $FL^2E^*_j \in \mathrm{Span}(LFLE^*_j, L^2FE^*_j, L^2E^*_j)$, $(3 \le j \le D)$.

(*ii*) $L^2FE^*_j \in \mathrm{Span}(LFLE^*_j, FL^2E^*_j, L^2E^*_j)$, $(2 \le j \le D-1)$.

(*iii*) $RL^2E^*_j \in \mathrm{Span}(LRLE^*_j, L^2RE^*_j, F^2LE^*_j, FLFE^*_j, LF^2E^*_j, LFE^*_j, FLE^*_j, LE^*_j)$, $(3 \le j \le D)$.

(*iv*) $L^2RE^*_j \in \mathrm{Span}(LRLE^*_j, RL^2E^*_j, F^2LE^*_j, FLFE^*_j, LF^2E^*_j, LFE^*_j, FLE^*_j, LE^*_j)$, $(2 \le j \le D)$.

Chapter 31

The “Inverse” of R

Wednesday, April 14, 1993

Let $\Gamma = (X, E)$ be any graph of diameter $D \geq 2$. Fix a vertex $x \in X$. Let $E_i^* \equiv E_i^*(x)$, and $T \equiv T(x)$.

Recall adjacency matrix

$$A = R + L + F \tag{31.1}$$

$$R = \sum_{i=0}^{D} E_{i+1}^* A E_i^*, \tag{31.2}$$

$$L = \sum_{i=0}^{D} E_{i-1}^* A E_i^*, \tag{31.3}$$

$$F = \sum_{i=0}^{D} E_i^* A E_i^*. \tag{31.4}$$

Observe R is not invertible (indeed $RE_D^* = O$.) So, R^{-1} does not exist.

Below we find a matrix “R^{-1}”$\in T(x)$ such that $R^{-1}Rv = v$ for “almost all” $v \in V$.

Lemma 31.1. *Let $\Gamma = (X, E)$ denote any graph, and the standard module V over $\mathbb{C}$.*

Fix a vertex $x \in X$, write

$$R \equiv R(x), \quad L \equiv L(x), \quad E_i^* \equiv E_i^*(x) \quad \textit{for all } i.$$

Then,

(i) There exists unique “R^{-1}”$\in \mathrm{Mat}_X(\mathbb{C})$ such that;

(ia) $R^{-1}v = 0$ *if* $Lv = 0$ *for* $v \in V$.

(ib) $R^{-1}RLv = Lv$ *for all* $v \in V$.

(ii) $R^{-1}(E_i^* V) \subseteq E_{i-1}^* V$ $(0 \leq i \leq D)$, $E_{-1}^* V = 0$.

(iii) $R^{-1} \in \mathrm{Mat}_X(\mathbb{Q})$.

(iv) $R^{-1} \in T(x)$.

Proof.

(i) Consider the orthogonal direct sum.

$$V = (\mathrm{Ker}L) + (\mathrm{Ker}L)^{\perp}.$$

Claim 1. $RL(\mathrm{Ker}L)^{\perp} \subseteq (\mathrm{Ker}L)^{\perp}$.

Proof of Claim 1. Pick $v \in (\mathrm{Ker}L)^{\perp}$, and $w \in \mathrm{Ker}L$. Show

$$\langle RLv, w\rangle = 0.$$

But

$$\bar{R}^{\top} = R^{\top} = \left(\sum_{i=0}^{D} E_{i+1}^* A E_i^*\right)^{\top} = \sum_{i=0}^{D} E_i^* A E_{i+1}^* = L.$$

So,

$$\langle RLv, w\rangle = \langle Lv, \bar{R}^{\top} w\rangle = \langle Lv, Lw\rangle = 0.$$

Claim 2. $RL : (\mathrm{Ker}L)^{\perp} \to (\mathrm{Ker}L)^{\perp}$ is an isomorphism of vector spaces.

Proof of Claim 2. It suffices to show above map is one-to-one.

Suppose there is a vector $v \in (\mathrm{Ker}L)^{\perp}$ such that $RLv = 0$.

Then,

$$0 = \langle RLv, v\rangle = \langle Lv, \bar{R}^{\top} v\rangle = \|Lv\|^2.$$

So $Lv = 0$.

Hence $v \in \mathrm{Ker}L \cap (\mathrm{Ker}L)^{\perp} = 0$.

This proves Claim 2.

Now "R^{-1} denote the unique matrix in $\mathrm{Mat}(\mathbb{C})$ such that

$$R^{-1}v = \begin{cases} 0 & \text{if } v \in \mathrm{Ker}L \\ L(RL)^{-1}v & \text{if } v \in (\mathrm{Ker}L)^{\perp}. \end{cases} \tag{31.5}$$

Observe that $(RL)^{-1} : (\mathrm{Ker}L)^{\perp} \to (\mathrm{Ker}L)^{\perp}$ exists by Claim 2.

Observe R^{-1} satisfies (ia) by (31.5).

Claim 3. R^{-1} satisfies (ib).

Proof of Claim 3. It suffices to check

$$R^{-1}RLv = Lv$$

for $v \in \mathrm{Ker}L$ and $v \in (\mathrm{Ker}L)^{\perp}$.

The case $v \in \mathrm{Ker}L$ is clear. So assume $v \in (\mathrm{Ker}L)^{\perp}$ by Claim 1. So,

$$R^{-1}(RLv) = L(RL)^{-1}RLv = Lv$$

as desired.

Uniqueness: Suppose a matrix $\hat{R}^{-1} \in \mathrm{Mat}_X(\mathbb{C})$ satisfies $(ia), (ib)$. Then, $\hat{R}^{-1}$ satisfies (31.5) above.

(*Pf.* The first part is clear. Let $v \in (\mathrm{Ker}L)^{\perp}$. By Claim 2, there exists $w \in (\mathrm{Ker}L)^{\perp}$ such that $v \in RLw$. So $\hat{R}^{-1}v = \hat{R}^{-1}RLw = Lw = L(RL)^{-1}v$.)

Therefore, $\hat{R}^{-1}$ agrees with R^{-1} on a basis for V, and $\hat{R}^{-1} = R^{-1}$.

(ii) Pick $v \in E_i^*V$. Show $R^{-1}v \in E_{i-1}^*V$.

Without loss of generality we may assume that $v \in \mathrm{Ker}L$ or $v \in (\mathrm{Ker}L)^{\perp}$.

If $v \in \mathrm{Ker}L$, then $R^{-1}v = 0 \in E_{i-1}^*V$.

If $v \in (\mathrm{Ker}L)^{\perp}$, then

$$R^{-1}v = L(RL)^{-1}v \in LE_i^*V \subseteq E_{i-1}^*V.$$

(iii) Observe $R, L \in \mathrm{Mat}_X(\mathbb{Q})$.

So V, $\mathrm{Ker}L$, each has basis consisting of vectors in $\mathbb{Q}^{|X|}$.

Replacing the construction of R^{-1} with the base field replaced by $\mathbb{Q}$, we find a matrix $\tilde{R}^{-1} \in \mathrm{Mat}_X(\mathbb{Q})$ satisfying (ia), (ib).

Now R^{-1} and $\tilde{R}^{-1}$ agree on a basis, and hence $R^{-1} = \tilde{R}^{-1}$.

(iv) $RL = \bar{L}^{\top}L$ is a real symmetric matrix. So it is diagonalizable.

Let θ be any eigenvalue of RL. Let V_θ denote the corresponding maximal eigenspace in V. Then

$$V = \sum_{\theta:\text{eigenvalue for } RL} V_\theta \quad \text{(orthogonal direct sum)}.$$

Let $E_\theta : V \to V_\theta$ denote the orthogonal projection. Then E_θ is a complex polynomial in RL.

Thus $E_\theta \in T(x)$.

HS MEMO

E_θ is real. Since RL is an integral matrix, every eigenvalue of RL is an algebraic integer.

Claim 4. We have

$$R^{-1} = \sum_{\theta:\text{eigenvalue of } RL} \theta^{-1} L E_\theta. \tag{31.6}$$

In particular, $R^{-1} \in T(x)$.

Proof of Claim 4. Show two sides of (31.6) agree, when applied to arbitrary $v \in V$.

Without loss of generality, we may assume that $v \in V_\theta$ for some eigenvalue θ of RL.

Let θ' denote any eigenvalue of RL.

$$E_{\theta'} v = \begin{cases} 0 & \text{if } \theta' \neq \theta, \\ v & \text{if } \theta' = \theta. \end{cases}$$

RHS of (31.6) applied to v equals

$$\begin{cases} 0 & \text{if } \theta = 0, \\ \theta^{-1} L v & \text{if } \theta \neq 0. \end{cases}$$

Show this equals $R^{-1} v$.

Case $\theta = 0$: Since $RLv = 0$,

$$0 = \langle v, RLv \rangle = \|Lv\|^2.$$

Hence $Lv = 0$, or $v \in \mathrm{Ker} L$. By (ia), $R^{-1} v = 0$.

Case $\theta \neq 0$: Since $RLv = \theta v$, $v = \theta^{-1} RLv$. Hence,

$$R^{-1} v = \theta^{-1} R^{-1} RLv = \theta^{-1} Lv$$

by (ib).

$\square$

Chapter 32

Irreducible Modules of Endpoint i

Monday, April 19, 1993

Lemma 32.1. *Let $\Gamma = (X, E)$ be any graph. With the notation of Lemma 31.1, the following hold.*

(i) Let W denote a thin irreducible T-module with endpoint r, diameter d. Pick i $(0 \le i \le d)$, and pick $v \in E^*_{r+i}W$. Then,

$$R^{-1}Rv = \begin{cases} v & \text{if } i < d, \\ 0 & \text{if } i = d. \end{cases}$$

(ii) Assume Γ is distance regular and thin with respect to x. Pick t $(0 \le i < D/2)$, and pick $v \in E^*_t V$. Then

$$R^{-1}R^i v = R^{i-1}v \quad (1 \le i \le D - 2t).$$

In particular, $R^{-1}Rv = v$.

(iii) Assume Γ is distance regular and thin with respect to x. Then

$$R : E^*_i V \to E^*_{i+1}V \quad (0 \le i < D/2)$$

is one-to-one.

Proof.

(i) Let $w_0, w_1, \ldots, w_d$ be a basis for W and $w_i \in E^*_{r+i}W$,

$$Rw_i = w_{i+1} \quad (0 \le i < d), \quad Lw_i = x_i(W)w_{i-1} \quad (1 \le i \le d).$$

So,

$$RLw_i = x_i(W)w_i \quad (1 \le i \le d).$$

(See Lemma 9.1.)

We want to find $R^{-1}Rw_i$.

If $i = d$, $R^{-1}Rw_d = 0$.

If $0 \le i < d$,

$$\begin{aligned} R^{-1}Rw_i &= R^{-1}w_{i+1} && (32.1)\\ &= x_{i+1}(W)^{-1}R^{-1}RLw_{i+1} && (32.2)\\ &= x_{i+1}(W)^{-1}Lw_{i+1} && (32.3)\\ &= x_{i+1}(W)^{-1}x_{i+1}(W)w_i && (32.4)\\ &= w_i. && (32.5) \end{aligned}$$

Thus, we have (i).

HS MEMO

$$\begin{aligned} RLw_i &= Rx_i(W)w_{i-1} = x_i(W)w_i, && (32.6)\\ LRw_i &= Lw_{i+1} = x_{i+1}(W)w_i, && (32.7)\\ [L,R]w_i &= (x_{i+1}(W) - x_i(W))w_i, \quad (0 \le i \le d), && (32.8)\\ & x_0(W) = 0, \quad x_{d+1}(W) = 0, && (32.9)\\ [L,R]|_W &= \sum_{i=0}^{d}(x_{i+1} - x_i(W))E^*_{r+i}|_W. && (32.10) \end{aligned}$$

(ii) Let

$$V = \sum W \quad \text{orthogonal direct sum of thin irreducible } T\text{-modules.}$$

Then,

$$E^*_tV = \sum_{r(W)\le t} E^*_tW \quad \text{(orthognal direct sum).}$$

Without loss of generality, we may assume

$$v \in E^*_tW$$

for some thin irreducible T-module with endpoint at most t.

Now if $i \leq D - 2t$, then

$$\begin{aligned} t + i &\leq D - t && (32.11)\\ &\leq D - r(W) && (32.12)\\ &\leq r(W) + d(W) \quad (D \leq 2r + d), && (32.13)\end{aligned}$$

by Lemma 14.1 (*iii*).

So

$$t + i - 1 \leq r(W) + d(W) - 1.$$

Hence,

$$\begin{aligned} R^{-1}R^i v &= R^{-1}R(R^{i-1}v) \quad (R^{i-1}v \in E^*_{t+i-1}W) && (32.14)\\ &= R^{i-1}v \quad \text{by } (i). && (32.15)\end{aligned}$$

(*iii*) Suppose $Rv = 0$ for some $v \in E^*_i V$ $(0 \leq i < D/2)$. Then

$$0 = R^{-1}Rv = v,$$

by (*ii*) with $t = i$ and $i = 1$.

□

Definition 32.1. Let $\Gamma = (X, E)$ denote any graph with the standard module V. Fix a vertex $x \in X$. Write $E^*_i \equiv E^*_i(x)$, $T \equiv T(x)$, $L \equiv L(x)$.

1. For every i $(0 \leq i \leq D)$, define subspace $V_i := V_i(x) \subseteq V$ by

$$V_i = \sum W,$$

   where the sum begin over irreducible T-modules W with endpoint i.

Observe:

$$V = V_0 + V_1 + \cdots + V_D \quad \text{(orthogonal direct sum.)}$$

V_0 is the trivial T-module.

2. $(E^*_i V)_{new} \equiv E^*_i V_i \quad (0 \leq i \leq D)$.

In general,

$$(E^*_i V)_{new} \subseteq \mathrm{Ker}L \cap E^*_i V \subseteq \mathrm{Ker}L \cap E^*_i V \subseteq \mathrm{Ker}(LE^*_i).$$

If each irreducible T-module with endpoint strictly less than i is thin,

$$(E^*_i V)_{new} = \mathrm{Ker}L \cap E^*_i V \subseteq \mathrm{Ker}(L \cdot E^*_i).$$

We have the assertion.

HS MEMO

$$E_i^* V = \sum_{j<i} V_j + V_i.$$

For V_j part, take $w_{i-j} \in W$ irreducible with endpoint $j < i$. Then,

$$Lw_{i-j} = x_{i-j}(W) w_{i-j-1} \neq 0,$$

and

$$L|_{\sum_{j<i} E_i^* V_j} : \sum_{j<i} E_i^* V_j \to V$$

is one to one.

Lemma 32.2. *Let $\Gamma = (X, E)$ be distance regular of diameter $D \geq 3$. Fix a vertex $x \in X$, $R \equiv R(x)$. $L \equiv L(x)$, $F \equiv F(x)$. Pick $v \in (E_1^* V)_{new}$. Then,*

(i) $RE_i^* A_{i-1} v = c_i E_{i+1}^* A_i v \quad (1 \leq i \leq D)$.

(ii) $FE_i^* A_{i-1} v = RE_{i-1}^* A_i V + (a_{i-1} - c_i + c_{i-1}) E_i^* A_{i-1} v + c_i E_i^* A_{i+1} v \ (1 \leq i \leq D)$.

(iii) $LE_i^* A_{i-1} v = FE_{i-1}^* A_i V + (a_{i-1} - c_i + c_{i-1}) E_{i-1}^* A_i v + b_{i-1} E_{i-1}^* A_{i-2} v \ (2 \leq i \leq D)$.

(iv) $LE_i^* A_{i+1} v = b_i E_{i-1}^* A_i v \quad (1 \leq i \leq D-1)$.

Proof.

(*i*) Let

$$v = \sum_{y \in X, \partial(x,y)=1} \alpha_y \hat{y} \quad \text{for some } \{\alpha_g\} \subseteq \mathbb{C}.$$

Then

$$Lv = \left(\sum_{y \in X, \partial(x,y)=1} \alpha_y \right) \hat{x} = 0.$$

So,

$$\sum_{y \in X, \partial(x,y)=1} \alpha_y = 0.$$

Thus,

$$v = \sum_{y \in X, \partial(x,y)=1} \alpha_y (\hat{y} - \hat{x}).$$

Let

$$\tilde{A}_i = A_0 + A_1 + \cdots + A_i \quad (0 \leq i \leq D).$$

Then

$$\tilde{A}_i v = \sum_{y\in X, \partial(x,y)=1} \alpha_y \tilde{A}_i(\hat{y} - \hat{x}) \tag{32.16}$$

$$= \sum_{y\in X, \partial(x,y)=1} \alpha_y \left(\sum_{z\in X, \partial(y,z)=i, \partial(x,z)=i+1} \hat{z} - \sum_{z'\in X, \partial(y,z')=i+1, \partial(x,z')=i} \hat{z}' \right) \tag{32.17}$$

$$= \sum_{y\in X, \partial(x,y)=1} \alpha_y (E^*_{i+1} A_i \hat{y} - E^*_i A_{i+1} \hat{y}) \tag{32.18}$$

$$= E^*_{i+1} A_i v - E^*_{i+1} A_{i+1} v. \tag{32.19}$$

Recall (Claim 1 in the proof of Theorem 16.1.)

$$A\tilde{A}_i = c_{i+1}\tilde{A}_{i+1} + (a_i - c_{i+1} + c_i)\tilde{A}_i + b_i\tilde{A}_{i-1} \quad (0 \le i \le D-1).$$

(This is valid for $i = 0$ as $A\tilde{A}_0 = AI = c_1\tilde{A} - \tilde{A}_0 = A$ by setting $\tilde{A}_{i-1} = O$.)

Now $(i) - (iv)$ are obtained by applying this to v on the right and multiplied by E^*_j $(0 \le j \le D)$ on the left.

□

HS MEMO

$A\tilde{A}_{i-1}v = AE^*_i A_{i-1} v - AE^*_{i-1} A_i v$. For $1 \le i \le D$,

$$(c_i\tilde{A}_i + (a_{i-1} - c_i + c_{i-1})\tilde{A}_{i-1} + b_{i-1}\tilde{A}_{i-2})v \tag{32.20}$$

$$= c_i E^*_{i+1} A_i v - c_i E^*_i A_{i+1} v \tag{32.21}$$

$$+ (a_{i-1} - c_i + c_{i-1}) E^*_i A_{i-1} v - (a_{i-1} - c_i + c_{i-1}) E^*_{i-1} A_i v \tag{32.22}$$

$$+ b_{i-1} E^*_{i-1} A_{i-2} v - b_{i-1} E^*_{i-2} A_{i-1} v. \tag{32.23}$$

(i) $RE^*_i A_{i-1} v = E^*_{i+1} A E^*_i A_{i-1} v = c_i E^*_{i+1} A_i v$ $(1 \le i \le D)$.

(ii) For $1 \le i \le D$,

$$FE^*_i A_{i-1} v = E^*_i A E^*_i A_{i-1} v \tag{32.24}$$

$$= RE^*_{i-1} A_i v - c_i E^*_i A_{i+1} v + (a_{i-1} - c_i + c_{i-1}) E^*_i A_{i-1} v. \tag{32.25}$$

(iii) For $2 \le i \le D$,

$$LE^*_i A_{i-1} v = E^*_{i-1} A E^*_i A_{i-1} v \tag{32.26}$$

$$= FE^*_{i-1} A_i v - (a_{i-1} - c_i + c_{i-1}) E^*_{i-1} A_i v + b_{i-1} E^*_{i-1} A_{i-2} v. \tag{32.27}$$

(Even if $i=1$, this is valid by setting $A_{i-2}=O$.)

(iv) For $1 \leq i \leq D-1$, $LE_i^*A_{i+1}v = E_{i-1}^*AE_i^*A_{i+1} = b_iE_{i-1}^*A_iv$.

Lemma 32.3. *Let $\Gamma=(X,E)$ be distance regular of diameter $D \geq 3$. Fix a vertex $x \in X$, $T \equiv T(x)$, $E_i^* \equiv E_i^*(x)$, $R=R(x)$, $F=F(x)$, $L=L(x)$.*

For every $v \in (E_1^*V)_{new}$, the following are equivalent.

(i) $E_i^*A_{i-1}v$, $E_i^*A_{i+1}v$ are linearly dependent for every i $(1 \leq i \leq D-1)$.

(ii) There exists a thin irreducible T-module W with endpoint 1 that contains v.

If (i), (ii) hold then

$$W = \mathrm{Span}(E_1^*A_0v, E_2^*A_1v, \dots, E_D^*A_{i-1}v).$$

Proof. $(ii) \to (i)$. Clear as

$$E_i^*A_{i-1}v,\ E_i^*A_{i+1}v \in E_i^*W = \mathrm{Span}(w_{i-1}).$$

$(i) \to (ii)$ Consider the sequence

$$E_1^*A_0v, E_2^*A_1v, E_3^*A_2v, \dots, E_{D+1}^*A_Dv.$$

The first term is nonzero and the last term is 0. So there exists

$$n := \min\{i \mid 1 \leq i \leq D,\ E_{i+1}^*A_iv = 0\}.$$

Now

$$E_{j+1}^*A_jv = 0 \quad (n \leq j \leq D). \tag{32.28}$$

HS MEMO

Use induction and Lemma 32.2 (i),

$$E_{j+1}^*A_jv \in \mathrm{Span}(RE_j^*A_{j-1}v) \quad (j \geq 1).$$

By our assumption (i), and the definition of n,

$$E_j^*A_{j+1}v \in \mathrm{Span}(E_j^*A_{j-1}v) \neq 0 \quad (1 \leq j \leq n).$$

By Lemma 32.2 (i),

$$RE_j^*A_{j-1}v \in \mathrm{Span}(E_{j+1}^*A_jv) \quad (1 \leq j \leq n).$$

By Lemma 32.2 (ii),

$$\begin{align} FE_j^*A_{j-1}v &\in \mathrm{Span}(RE_{j-1}^*A_jv, E_j^*A_{j-1}v, E_j^*A_{j+1}v) \tag{32.29}\\ &\subseteq \mathrm{Span}(RE_{j-1}^*A_{j-2}v, E_j^*A_{j-1}v) \tag{32.30}\\ &\mathrm{Span}(E_{j-1}^*A_{j-1}v) \quad (1 \leq j \leq n). \tag{32.31} \end{align}$$

By Lemma 32.2 (*iii*),

$$\begin{aligned} FE^*_j A_{j-1} v &\in \mathrm{Span}(FE^*_{j-1} A_j v, E^*_{j-1} A_j v, E^*_{j-1} A_{j-2} v) && (32.32)\\ &\subseteq \mathrm{Span}(FE^*_{j-1} A_{j-2} v, E^*_{j-1} A_{j-2} v) && (32.33)\\ &\subseteq \mathrm{Span}(E^*_{j-1} A_{j-2} v) \quad (2 \leq j \leq n). && (32.34) \end{aligned}$$

Hence,

$$W = \mathrm{Span}(E^*_1 A_0 v, E^*_2 A_1 v, \dots, E^*_n A_{n-1} v).$$

is R, F, L invariant.

Therefore W is a thin T-module with endpoint 1 that contains v. $\square$

Chapter 33

Algebra on First Subconstitutent

Wednesday, April 21, 1993

Lemma 33.1. *Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ be a commutative scheme. Fix a vertex $x \in X$, write $E_i^* \equiv E_i^*(x)$, $M^* \equiv M^*(x)$, $T \equiv T(x)$. Then the following hold.*

(i) $E_0^* M M^* = E_0^* M$

(ii) $E_0^* T = E_0^* M$.

(iii) $T E_0^* T = M E_0^* M$.

(iv) $E_0^* E_0 E_0^* = |X|^{-1} E_0^*$.

(v) $E_0^* E_0 E_0^* = |X|^{-1} E_0^*$.

(vi) Lines (i)-(iv) hold if we interchange (E_0, E_0^*), (M, M^*).

Moreover, $M E_0^* M = M^* E_0 M^*$.

Proof.

(i) $\supseteq$: $1 \in M^*$ implies $M \subseteq M M^*$.

$\subseteq$: Pick $\alpha \in E_0^* M M^*$. Show $\alpha \in E_0^* M$. Since $A_0, A_1, \dots, A_D$ span M, and since $E_0^*, E_1^*, \dots, E_D^*$ span M^*, without loss of generality we may assume that

$$\alpha = E_0^* A_i E_j^*$$

for some $i, j \in \{0, \dots, D\}$.

Without loss of generality we may assume taht $i = j$, else $\alpha = 0$ by Lemma 20.3.

$$(E_h^* A_i E_j^* \neq O \Leftrightarrow p_{hi}^j \neq 0.)$$

Now

$$\alpha = E_0^* A_i \left(\sum_{h=0}^{D} E_h^* \right) = E_0^* A_i \in E_0^* M.$$

(ii) $\supseteq$: This is clear.

$\subseteq$: $E_0^* T$ is the minimal right ideal of T containing E_0^*.

So, we just have to show that $E_0^* M$ is a right ideal of T containing E_0^*.

It clearly contains E_0^* since $I \in M$, and is a right ideal of T by (i), and the fact that T is generated by M abd M^*.

(iii) By the transpose of (ii),

$$TE_0^* = ME_0^*,$$

so,

$$TE_0^* T = (TE_0^*)(E_0^* T) = ME_0^* E_0^* M = ME_0^* M.$$

(iv) We have

$$E_0^* E_0 E_0^* = \frac{1}{|X|} E_0^* \left(\sum_{h=0}^{D} A_h \right) E_0^* = \frac{1}{|X|} E_0^* A_0 E_0^* = |X|^{-1} E_0^*.$$

(v) The first part is clear by using Lemma 20.3 (ii),

$$E_h A_i^* E_j \neq O \Leftrightarrow q_{hi}^j \neq 0,$$

and Lemma 22.1 (iii),

$$q_{0i}^j = \delta_{ij}.$$

Also,

$$ME_0^* M = TE_0^* T = TE_0^* E_0 E_0^* T \subseteq TE_0 T = M^* E_0 M^*,$$

and

$$M^* E_0 M^* \subseteq ME_0^* M$$

by dual argument. So,

$$M^* E_0 M^* = ME_0^* M.$$

This proves the lemma.

$\square$

Lemma 33.2. *Let $\Gamma = (X, E)$ be distance regular of diameter $D \leq 3$, Q-polynomial with respect to $E_0, E_1, \ldots, E_D$. Pick a vertex $x \in X$, write $E_i^* \equiv E_i^*(x)$, $M^* \equiv M^*(x)$, $T \equiv T(x)$.*

(i) $E_1^* M M^* = E^* M + E_1^* E_0 M^* + E_1^* E_1 M^*$.

(ii) $E_1 M^* M = E_1 M^* + E_1 E_0^* M + E_1 E_1^* M$.

Proof.

(i) View E_{-1}^*, E_{D+1}^* as O.

View θ_{-1}^*, θ_{D+1}^* as indeterminates.

Let Δ denote RHS in (i).

$\supseteq$: $I \in M^*$ implies $M \subseteq MM^*$.

$\subseteq$: Suppose not. Then there exists

$$\alpha \in E_1^* M M^* \quad \Delta. \tag{33.1}$$

Since $A_0, A_1, \ldots, A_D$ span M, since $E_0^*, E_1^*, \ldots, E_D^*$ span M^*, without loss of generality we may assume that

$$\alpha = E_1^* A_i E_j^*$$

for some $i, j \in \{0, \ldots, D\}$.

Observe $|i - j| \leq 1$, else $\alpha = 0$ by Lemma 20.3.

Without loss of generality, assume $i + j$ is minimal subject to the above constraints.

First assume

$$j = i + 1. \tag{33.2}$$

Observe

$$\begin{aligned}
& E_1^* A_i E_{i+1}^* + E_1^* A_i E_i^* + E_1^* A_i E_{i-1}^* && (33.3)\\
&= E_1^* A_i \left(\sum_{h=0}^{D} E_h^* \right) && (33.4)\\
&= E_1^* A_i && (33.5)\\
&\in \Delta. && (33.6)
\end{aligned}$$

Also, observe

$$E_1^* A_i E_i^*,\ E_1^* A_i E_{i-1}^* \in \Delta$$

by the minimality of $i + j$, so

$$\alpha = E_1^* A_i E_{i+1}^* \in \Delta$$

by (33.6). Hence, (33.2) cannot occur.

Since $|i-j| \leq 1$,

$$i \in \{j, j+1\}. \tag{33.7}$$

Observe

$$\begin{aligned} & E_1^* A_{j+1} E_j^* + E_1^* A_j E_j^* + E_1^* A_{j-1} E_j^* && (33.8) \\ & = E_1^* \left(\sum_{h=0}^{D} A_h \right) E_j^* && (33.9) \\ & = |X| E_1^* E_0 E_j^* && (33.10) \\ & \in \Delta, && (33.11) \end{aligned}$$

and

$$\begin{aligned} & \theta_{j+1}^* E_1^* A_{j+1} E_j^* + \theta_j^* E_1^* A_j E_j^* + \theta_{j-1}^* E_1^* A_{j-1} E_j^* && (33.12) \\ & = E_1^* \left(\sum_{h=0}^{D} \theta_h^* A_h \right) E_j^* && (33.13) \\ & = |X| E_1^* E_1 E_j^* && (33.14) \\ & \in \Delta. && (33.15) \end{aligned}$$

Since $E_1^* A_{j-1} E_j^* \in \Delta$ by the minimality of $i+j$, so

$$E_1 A_{j+1} E_j^* + E_1^* A_j E_j^* \in \Delta,$$

$$\theta_{j+1}^* E_1^* A_{j+1} E_{j+1}^* + \theta_j^* E_1^* A_j E_j^* \in \Delta.$$

But, $\theta_0^*, \theta_1^*, \dots, \theta_D^*$ are distinct by Lemma 22.2 (iv), so

$$E_1^* A_{j+1} E_j^*,\ E_1^* A_j E_j^* \in \Delta.$$

But α is one of these two matrices, so

$$\alpha \in \Delta.$$

Hence, (33.7) cannot occur either, and we have a contradiction.

(ii) Dual argument.

□

Lemma 33.3. *With the above notation, set*

$$\tilde{J} := E_1^* J E_1^*, \quad \tilde{A} := E_1^* A E_1^*.$$

(i) $\tilde{J}^2 = k\tilde{J}$. *(k = valency of Γ)*

(ii) $\tilde{J}\tilde{A} = \tilde{A}\tilde{J} = a_1\tilde{J}$. $(a_1 = p^1_{11}$ *for* $\Gamma)$

(iii) $E^*_1 E_0 E^*_1 = |X|^{-1}\tilde{J}$.

(iv) $E^*_1 E_1 E^*_1 = |X|^{-1}(E^*_1(\theta^*_0 - \theta^*_2) + \tilde{A}(\theta^*_1 - \theta^*_2) + \tilde{J}(\theta^*_2))$.

Proof.

(i) The first subconstituent has k vertices.

(ii) The first subconstituent is regular of valency a_1.

(iii) Since $E_0 = |X|^{-1}J$,

$$E^*_1 E_0 E^*_1 = |X|^{-1}\tilde{J}.$$

(iv) We have

$$\begin{aligned} E^*_1 E_1 E^*_1 &= E^*_1 \left(|X|^{-1} \sum_{h=0}^{D} \theta^*_h A_h \right) E^*_1 && (33.16)\\ &= |X|^{-1}(\theta^*_0 E^*_1 A_0 E^*_1 + \theta^*_1 E^*_1 A_1 E^*_1 + \theta^*_2 E^*_1 A_2 E^*_1) && (33.17)\\ &= |X|^{-1}(\theta^*_0 E^*_1 + \theta^*_1 \tilde{A} + \theta^*_2 E^*_1 A_2 E^*_1). && (33.18) \end{aligned}$$

Also,

$$\begin{aligned} \tilde{J} &= E^*_1 J E^*_1 && (33.19)\\ &= E^*_1 A_0 E^*_1 + E^*_1 A_1 E^*_1 + E^*_1 A_2 E^*_1 && (33.20)\\ &= E^*_1 + \tilde{A} + E^*_1 A_2 E^*_1. && (33.21) \end{aligned}$$

Eliminating the $E^*_1 A_2 E^*_1$ term in (33.18) using equation (33.21), we get (iv).

□

Lemma 33.4. *With the above notation,*

(i) $E^*_1 T = E^*_1 E_0 M^* + E^*_1 M + E^*_1 E_1 M^* + E^*_1 E_1 E^*_1 M + \cdots$.

(ii) $E^*_1 T E^*_1 = \mathrm{Span}(E^*_1 E_0 E^*_1, E^*_1, E^*_1 E_1 E^*_1, (E^*_1 E_1 E^*_1)^2, \ldots)$.

(iii) $E^*_1 T E^*_1 = \mathrm{Span}(\tilde{J}, E^*_1, \tilde{A}, \tilde{A}^2, \ldots)$.

(iv) $E^*_1 T E^*_1$ *is symmetric (in particular, commutative).*

Proof.

(i) $\supseteq$: Clear.

$\subseteq$: $E^*_1 T$ is the minimal right ideal of Γ that contains E^*_1.

RHS contains E_1^*, so show RHS is a right ideal of T.

Show RHS is closed with respect to multiplication on right by M, M^*.

We have

$$E_1^*E_0M^*(M) = E_1^*E_0M^*,\ E_1^*E_0M^*(M^*) = E_1^*E_0M^*$$

by dual of Lemma 33.1 (i).

By Lemma 33.2,

$$\begin{aligned}
&E_1^*E_1E_1^*\cdots E_1^*M(M^*) && (33.22)\\
&\quad = E_1^*E_1E_1^*\cdots E_1(E_1^*MM^*) && (33.23)\\
&\quad = E_1^*E_1E_1^*\cdots E_1(E_1^*M + E_1^*E_0M^* + E_1^*E_1M^*) && (33.24)\\
&\quad \in \text{RHS}, && (33.25)
\end{aligned}$$

because

$$E_1^*E_1E_1^*\cdots E_1E_1^*E_0M^* \subseteq E_1^*TE_0T = E_1^*M^*E_0M^* = E_1^*E_0M^*.$$

By Lemma 33.2,

$$\begin{aligned}
&E_1^*E_1E_1^*\cdots E_1M^*(M) && (33.26)\\
&\quad = E_1^*E_1E_1^*\cdots E_1^*({}_1M^*M) && (33.27)\\
&\quad = E_1^*E_1E_1^*\cdots E_1^*(E_1M^* + E_1E_0^*M^* + E_1E_1^*M) && (33.28)\\
&\quad \in \text{RHS}, && (33.29)
\end{aligned}$$

because by the last part of Lemma 33.1,

$$E_1^*E_1E_1^*\cdots E_1^*E_1E_0^*M \subseteq E_1^*TE_0^*T = E_1^*ME_0^*M = E_1^*E_0M^*.$$

(ii) Multiply (i) on the right by E_1^*, we have

$$\begin{aligned}
E_1^*TE_1^* &= E_1^*E_0M^*E_1^* + E_1^*ME_1^* + E_1^*E_1M^*E_1^* && (33.30)\\
&\quad + \cdots + E_1^*E_1\cdots E_1M^*E_1^* + E_1^*E_1\cdots E_1^*ME_1^* && (33.31)\\
&= \text{Span}(E_1^*E_0E_1^*, E_1^*, E_1^*E_1E_1^*, (E_1^*E_1E_1^*)^2, \ldots). && (33.32)
\end{aligned}$$

HS MEMO

Note that by Lemma 29.1,

$$\begin{aligned}
E_1^*ME_1^* &= \text{Span}(E_1^*A_0E_1^*, E_1^*A_1E_1^*, E_1^*A_2E_1^*) && (33.33)\\
&= \text{Span}(E_1^*, E_1^*E_1E_1^*, E_1^*E_0E_1^*). && (33.34)
\end{aligned}$$

Moreover,

$$E_1^*\cdots E_1^*E_0E_1^* \subseteq E_1^*TE_0TE_1^* = E_1^*M^*E_0M^*E_1^* \in \text{Span}(E_1^*E_0E_1^*).$$

(iii) By (ii), $E_1^*TE_1^*$ is generated by $\tilde{J} = |X|E_1^*E_0E_1^*$ and $E_1^*E_1E_1^*$.

By Lemma 33.3 (iv), $E_1^*TE_1^*$ is generated by $\tilde{J}, \tilde{A}$.

But, Span$\tilde{J}$ is a 2-sided ideal by Lemma 33.3 (i), (ii).

Hence, we have (iii).

(iv) $\tilde{A}$, $\tilde{J}$ are symmetric commuting matrices, we have the claim.

$\square$

Chapter 34

Modules of Endpoint One

Friday, April 23, 1993

Let $\Gamma = (X, E)$ be distance-regular of diameter $D \geq 3$.

Assume Γ is Q-polynomial with respect to $E_0, E_1, \dots, E_D$. Write

$$\tilde{A}_i = A_0 + A_1 + \cdots + A_i \quad i \in \{0, 1 \dots, D\}.$$

Fix a vertex $x \in X$, write $E_i^* \equiv E_i^*(x)$, $M^* \equiv M^*(x)$, $T \equiv T(x)$.

Pick $0 \neq v \in (E_1^* V)_{new}$. Set $v^* = |X| E_1 v$. We will show that

$$Tv = Mv + M^* v^*.$$

We need a preliminary lemma.

Lemma 34.1. *With the atove notation, we have the following.*

(i) $\tilde{A}_h v = E_{h+1}^* A_h v - E_h^* A_{h+1} v$, $h \in \{0, 1, \dots, D\}$.

$(E_{D+1}^* = A_{D+1} = O)$.

(ii) $E_h^* v^* = (\theta_{h-1}^* - \theta_h^*) E_h^* A_{h-1} v - (\theta_h^* - \theta_{h+1}^*) E_h^* A_{h+1} v$, $h \in \{0, 1, \dots, D\}$. $(A_{-1} = A_{D+1} = O)$.

(iii) $(\theta_i^* - \theta_{i+1}^*) E_{i+1}^* A_i v = \left(\sum_{h=0}^{i} (\theta_h^* - \theta_{i+1}^*) A_h \right) v - \left(\sum_{h=0}^{i} E_h^* \right) v^*$, $i \in \{0, 1, \dots, D-1\}$.

(iv) $(\theta_i^* - \theta_{i+1}^*) E_i^* A_{i+1} v = \left(\sum_{h=0}^{i-1} (\theta_h^* - \theta_i^*) A_h \right) v - \left(\sum_{h=0}^{i} E_h^* \right) v^*$, $i \in \{0, 1, \dots, D-1\}$.

(v) $Mv + M^* v^* = \mathrm{Span}\{E_i^* A_{i-1} v, E_{i-1}^* A_i v \mid 1 \leq i \leq D\}$.

Proof.

(i) It is already done in Lemma 32.2.

(ii)

$$\begin{aligned}
E_h^* v^* &= |X| E_h^* E_1 v && (34.1)\\
&= E_h^* \left(\sum_{i=0}^{D} \theta^* A_i \right) v && (34.2)\\
&= E_h^* \left(\sum_{i=0}^{D} \theta_i^* (\tilde{A}_i - \tilde{A}_{i-1}) \right) v && (34.3)\\
&= E_h^* \left(\sum_{i=0}^{D-1} (\theta_i^* - \theta_{i+1}^*) \tilde{A}_i \right) v + E_h^* \theta_D^* \tilde{A}_D v && (34.4)\\
&= E_h^* \left(\sum_{i=0}^{D-1} (\theta_i^* - \theta_{i+1}^*)(E_{i+1}^* A_i v - E_i^* A_{i+1} v) \right) && (34.5)\\
&= (\theta_{h-1}^* - \theta_h^*) E_h^* A_{h-1} v - (\theta_h^* - \theta_{h+1}^*) E_h^* A_{h+1} v. && (34.6)
\end{aligned}$$

(iii), (iv) Call the equation in (iii), i^+ and call the equation in (iv) i^-. Prove in order,

$$0^-, 0^+, 1^-, 1^+, 2^-, 2^+, \ldots.$$

0^-: Trivial.

HS MEMO

$$\begin{aligned}
\text{LHS} &= (\theta_0^* - \theta_1^*) E_0^* A_1 v && (34.7)\\
&= (\theta_{-1}^* - \theta_1^*) E_0^* A_{-1} v - E_h^* v^* \quad (\text{by } (ii)) && (34.8)\\
&= -E_0^* v^* && (34.9)\\
&= \text{RHS}. && (34.10)
\end{aligned}$$

i^+: using (i) and i^-.

$$\begin{aligned}
\text{LHS} &= (\theta_i^* - \theta_{i+1}^*) E_{i+1}^* A_i v && (34.11)\\
&= (\theta_i^* - \theta_{i+1}^*) E_i^* A_{i+1} v + (\theta_i^* - \theta_{i+1}^*) \tilde{A}_i v \quad (\text{by } (i)) && (34.12)\\
&= \left(\sum_{h=0}^{i-1} (\theta_h^* - \theta_i^*) A_h \right) v - \left(\sum_{h=0}^{i} E_h^* \right) v^* + (\theta_i^* - \theta_{i+1}^*) \left(\sum_{h=0}^{i} A_h \right) v \quad (\text{by } i^-) && (34.13)\\
&= \left(\sum_{h=0}^{i} (\theta_h^* - \theta_{i+1}^*) A_h \right) v - \left(\sum_{h=0}^{i} E_h^* \right) v^*. && (34.14)
\end{aligned}$$

i^-: using (ii) and $(i-1)^+$.

$$\begin{aligned}
\text{LHS} &= (\theta^*_i - \theta^*_{i+1})E^*_i A_{i+1} v && (34.15)\\
&= (\theta^*_{i-1} - \theta^*_i)E^*_i A_{i-1} v - E^*_i v^* \quad \text{(by } (ii)) && (34.16)\\
&= \left(\sum_{h=0}^{i-1} (\theta^*_h - \theta^*_i) A_h\right) v - \left(\sum_{h=0}^{i-1} E^*_h\right) v^* - E^*_i v^* && (34.17)\\
&= \left(\sum_{h=0}^{i-1} (\theta^*_h - \theta^*_i) A_h\right) v - \left(\sum_{h=0}^{i} E^*_h\right) v^*. && (34.18)
\end{aligned}$$

(v) Immediate from $(i) - (iv)$.

$\square$

HS MEMO

$$\begin{aligned}
Mv + M^* v^* &\subseteq \text{Span}\{\tilde{A}_h v, E^*_h v^* \mid 0 \le h \le D\} && (34.19)\\
&\subseteq \text{Span}\{E^*_h A_{h-1} v, E^*_{h-1} A_h v \mid 1 \le h \le D\} && (34.20)
\end{aligned}$$

by (i) and (ii).

On the other hand,

$$E^* h A_{h-1} v, E^*_{h-1} A_h v \in Mv + M^* v^* \quad i \in \{1, 2, \dots, D\}$$

by (iii) and (iv).

Lemma 34.2. *With the notation of Lemma 34.1, assume $0 \neq v \in (E^*_1 V)_{new}$ is an eigenvector for $\tilde{A} := E^*_1 A E^*_1$. Then*

(i) $Tv = Mv + M^* v$, where $v^* = |X| E_1 v$.

(ii) $Tv = \text{Span}\{v^+_1, v^+_2, \dots, v^+_D, v^-_2, v^-_3, \dots, v^-_{D-1}\}$, where $v^+_i = E^*_i A_{i-1} v$, $v^-_i = E^*_i A_{i+1} v$.

(iii) $\dim E^*_1 Tv = 1$, $\dim E^*_i Tv \le 2$ for $i \in \{2, \dots, D-1\}$, and $\dim E^*_D Tv \le 1$.

(iv) Tv is an irreducible T-module.

Proof.

(i) $\supseteq$: $v \in Tv$. So $Mv \subseteq Tv$, and

$$v^* \in Mv \subseteq Tv.$$

Hence, $M^* v^* \subseteq Tv$.

$\subseteq$: It suffices to show that $Mv + M^* v^*$ is a T-module (since it clearly contains v).

Show:

(a) $M^*Mv \subseteq Mv + M^*v^*$.

(b) $MM^*v \subseteq Mv + M^*v^*$.

Proof of (a). By the transpose of (i) in Lemma 33.2,

$$M^*ME_1^* = ME_1^* + M^*E_0E_1^* + M^*E_1E_1^*.$$

Since $v \in E_1^*V$, $E_1^*v = v$ and

$$M^*Mv = Mv + M^*E_0v + M^*E_1v.$$

But also $E_0v = 0$ since v is orthogonal to the trivial T-module. Since $E_1v = |X|^{-1}v^*$,

$$M^*Mv = Mv + M^*v^*$$

as desired.

(b) is obtained from the traspose of (ii) in Lemma 33.2.

HS MEMO

$$\begin{aligned} MM^*v &= MM^*E_1v^* && (34.21)\\ &= M^*E_1v^* + ME_0^*E_1v^* + ME_1^*E_1v^* && (34.22)\\ &= M^*v^* + ME_0^*v^* + ME_1^*v^*. && (34.23) \end{aligned}$$

$E_0^*v^* \in Tv$ and $E_0^*Tv = 0$ as $v \in (E_1^*V)_{new}$. So, $E_0^*v^* = 0$.

$$\begin{aligned} E_1^*v^* &= |X|E_1^*E_1v && (34.24)\\ &= |X|E_1^*E_1E_1^*v && (34.25)\\ &= ((\theta_0^* - \theta_2^*)E_1^* + (\theta_1^* - \theta_2^*)E_1^*AE_1^* + \theta_2^*|X|E_1^*E_0E_1^*)v && (34.26)\\ &= (\theta_0^* - \theta_2^*)v + (\theta_1^* - \theta_2^*)E_1^*AE_1^*v + \theta_2^*|X|E_1^*E_0v && (34.27)\\ &\in \mathrm{Span}\{v\}, && (34.28) \end{aligned}$$

as $E_0v = 0$, and v is an eigenvector of $E_1^*AE_1^*$.

$\star$ $v \in (E_1^*V)_{new}$. If v is an eigenvector of $E_1^*AE_1^*$,

$$E_1^*v^* \in \mathrm{Span}\{v\}.$$

(ii) We have

$$\begin{aligned} Tv &= Mv + M^*v^* && (34.29)\\ &= \mathrm{Span}\{E_i^*A_{i-1}v, E_{i-1}^*A_iv \mid 1 \le i \le D\} && (34.30)\\ &= \mathrm{Span}\{v_i^+, v_{i-1}^- \mid 1 \le i \le D\} && (34.31)\\ &= \mathrm{Span}\{v_1^+, v_2^+, \dots, v_D^*, v_0^-, \dots, v_{D-1}^-\} && (34.32) \end{aligned}$$

by Lemma 34.1 (v).

But $v_0^- = E_0^* A_1 v = 0$ since $v \in (E_1^*V)_{new}$, and $v_1^- \in \mathrm{Span}\{v_1^+\}$.

Indeed,

$$v_1^- = E_1^* A_2 v = (-1 - a_0(Tv))v_1^+.$$

where $a_0(Tv)$ is the eigenvalue of v associated with $\tilde{A}$.

To see this, observe

$$\begin{aligned} 0 &= \tilde{J}v && (34.33)\\ &= E_1^* \left(\sum_{i=0}^{D} A_i\right) E_1^* v && (34.34)\\ &= E_1^* \left(\sum_{i=0}^{2} A_i\right) E_1^* v && (34.35)\\ &= v + a_0(Tv)v + v_1^-. && (34.36)\end{aligned}$$

Therefore,

$$Tv = \mathrm{Span}\{v_1^+, v_2^+, \dots, v_D^+, v_2^-, \dots, v_{D-1}^-\}.$$

(iii) $v_i^+, v_i^- \in E_i^* V$.

(iv) Suppose Tv is reducible, i.e., $Tv = W_1 + W_2$. (orthogonal direct sum of nonzero T-modules)

$$E_1^* Tv = E_1^* W_1 + E_1^* W_2$$

has dimension 1 by *(iii)*. Assume $v \in E_1^* W_1$. Then $Tv \subseteq W_1$, a contradiction.

□

Lemma 34.3. *With the notation of Lemma 34.1, assume $0 \neq v \in (E_1^*V)_{new}$ is an eigenvector for $\tilde{A} := E_1^* A E_1^*$.*

(i) Tv is thin if and only if $M^* v^* \subseteq Mv$.

(ii) Let W denote any irreducible T-module with endpoint 1. Then

$$W = Tv'$$

for some $0 \neq v' \in (E_1^*V)_{new}$ that is an eigenvector of $\tilde{A}$.

(iii) Denote eigenvalue of $\tilde{A}$ associated to v (resp. v') by $a_0(Tv)$ (resp. $a_0(Tv')$).

Then Tv, Tv' are isomorphic T-module if and only if $a_0(Tv) = a_0(Tv')$.

(iv) $E_1^* T E_1^*$ has basis

$$\tilde{J}, E_1^*, \tilde{A}, \tilde{A}^2, \dots, \tilde{A}^{\ell-1},$$

where ℓ is the number of mutually nonisomorphic T-modules with endpoint 1.

Proof.

(i) If Tv is thin, then by Lemma 9.1, $Tv = Mv$. Hence $M^*v^* \subseteq Mv$.

HS MEMO

Originally, the statement was Tv is thin if and only if $M^*v^* = Mv$. This is not the case in general. Suppose Γ is thin. Let W be an irreducible T-module of endpoint 1. Then, that $W \cap E_1^*V \ni v \neq 0$ implies $v^* \in W \cap E_1V$ gives one to one and $k \leq m$.

However, by 'Distance-Regular Graphs' (A.E. Brouwer, 1989),

$J(v,d)$: $v \geq 2d$

$$b_j = (d-j)(v-d-j), \quad c_j = j^2, \tag{34.37}$$

$$\theta_j = (d-j)(v-d-j) - j, \quad m_j = \binom{v}{j} - \binom{v}{j-1}. \tag{34.38}$$

In particular,

$$k = b_0 = d(v-d) > m_1 = v-1 \quad \text{ if } d \geq 2,$$

and $J(v,d)$ is thin.

So $|X|E_1v = v^*$ may be 0 sometimes. But as Tv is dual thin of diameter at least $D-2$. The dual endpint $r^* \leq 2$, so in that case, $E_2v \neq 0$. Hence, if $D \geq 3$, $E_2v \neq 0$ always.

HS MEMO

Now assume $M^*v^* \subseteq Mv = Tv$. Then

$$Mv = \{f(A)v \mid f(\lambda) \in \mathbb{C}[\lambda]\}.$$

So,

$$E_iTv = E_iMv \in \mathrm{Span}(E_iv).$$

Hence, Tv is dual thin.

Now we can construct a basis, $0 \neq w_0^* \in E_{r^*}W$, where r^* is the dual endpoint, and

$$w_0^*, w_1^*, \dots, w_d^* \in W = Tv,$$

where $w_i^* = E^*_{r^*+i}{A_1^*}^i w_0^*$.

$$A_1^* w_i^* = w_{i+1}^* + a_i^* w_i^* + x_i^* w_{i-1}^*,$$

and $w_i^* = p_i^*(A^*)w_0^*$.

$$E^*_{r^*+i}A^*_1E_{r^*+i}|_{E_{r^*+i}W} = a_i^* \cdot 1|_{E_{r^*+i}W},$$

$$E^*_{r^*+i-1}A^*_1E_{r^*+i}A^*E_{r^*+i-1}|_{E_{r^*+i-1}W} = x_i^* \cdot 1|_{E_{r^*+i-1}W}.$$

See Lemma 9.1, and Lemma 22.2.

From above, $Tv = M^*w_0^*$. So,

$$E_i^*Tv = E_i^*M^*w_0^* \in \mathrm{Span}\{E_i^*w_0^*\}.$$

Thus, Tv is thin.

$*$Need to write down the dual at least for Lemma 9.1, Corollary 9.1.

(iii) E_1^*W is an $\tilde{A}$-module. So, there exists $0 \neq v' \in E_1^*W$ that is an eigenvalue for $\tilde{A}$. Also $Tv' \subseteq W$.

Since W is irreducible, $Tv' = W$.

(iii) Suppose $Tv \to Tv'$ is an isomorsphism of T-modules.

Recall $\sigma s = s\sigma$ for all $s \in T$.

$$\mathrm{Span}\{\sigma v\} = \sigma E_1^*Tv = E_1^*\sigma Tv = E_1^*Tv' = \mathrm{Span}\{v'\}.$$

Hence,

$$a_0(Tv)\sigma v = \sigma(a_0(Tv)v) = \sigma\tilde{A}v = \tilde{A}\sigma v = a_0(Tv')\sigma v.$$

Since $\sigma v \neq 0$, $a_0(Tv) = a_0(Tv')$.

Now suppose $a_0(Tv) = a_0(Tv')$. Show

$$\sigma : Tv \to Tv' \quad (sv \mapsto sv') \quad (s \in T)$$

is an isomorphism of T-modules.

Pick $s \in T$. Require $sv = 0$ if and only if $sv' = 0$.

Without loss of generality , $s \in TE_1^*$, since $v, v' \in E_1^*V$.

Now $0 = sv$ if and only if

$$0 = \|sv\|^2 = \bar{v}^\top \bar{s}^\top sv.$$

But, $\bar{s}^\top s \in E_1^*TE_1^*$.

Hence, by Lemma 33.4 (iii),

$$\bar{s}^\top s = \alpha\tilde{J} + p(\tilde{A})$$

for some $\alpha \in \mathbb{C}$ and $p(\lambda) \in \mathbb{C}[\lambda]$.

Thus, using the fact that $\tilde{J}v = 0$,

$$0 = \|sv\|^2 = \bar{v}^\top(\alpha\tilde{J} + p(\tilde{A}))v = \|v\|^2 p(a_0(Tv))$$

if and only if $0 = p(a_0(Tv))$.

Replacing v by v', we have

$$\begin{aligned} 0 = sv' &\leftrightarrow 0 = p(a_0(Tv')) && (34.39)\\ &\leftrightarrow 0 = p(a_0(Tv)) && (34.40)\\ &\leftrightarrow 0 = sv && (34.41) \end{aligned}$$

as desired.

(iv) The following hold.

$$\begin{aligned} \ell &= \text{the number of mutually nonisomorphic } T\text{-modules with endpoint } 1 && (34.42)\\ &= \text{the number of distinct eigenvalues of } \tilde{A} : (E_1^*V)_{new} \to (E_1^*V)_{new} && (34.43)\\ &= \text{the degree of minimal polynomial of } \tilde{A} : (E_1^*V)_{new} \to (E_1^*V)_{new}. && (34.44) \end{aligned}$$

Claim 1. $\tilde{J}.E_1^*, \tilde{A}, \dots, \tilde{A}^{\ell-1}$ are linearly independent.

Proof of Claim 1. Suppose not. Then

$$\alpha\tilde{J} + p(\tilde{A}) = O$$

for some $\alpha \in$ C and $p(\lambda) \in \mathbb{C}[\lambda]$ with $\deg p \le \ell - 1$.

But $\tilde{J}|_{(E_1^*V)_{new}} = O$ impiles $p(\tilde{A})|_{(E_1^*V)_{new}} = O$.

Since

$$\deg p < \text{the degree of minimal polynomial of } \tilde{A}|_{(E_1^*V)_{new}},$$

we find p is identically 0.

Then α is identically 0 also.

Claim 2. $\tilde{J}.E_1^*, \tilde{A}, \dots, \tilde{A}^{\ell-1}$ span $E_i^*TE_i^*$.

Proof of Claim 2. It needs to show

$$\tilde{J}.E_1^*, \tilde{A}, \dots, \tilde{A}^{\ell} \text{ are linearly dependent.} \tag{34.45}$$

Let m denote the minimal polynomial of $\tilde{A}|_{(E_1^*V)_{new}}$. So,

$$m(\tilde{A}|_{(E_1^*V)_{new}}) = 0.$$

Observe that

$$E_1^*V = (E_1^*V)_{new} + \mathrm{Span}\{A\hat{x}\}.$$

(direct sum of $E_1^*TE_1^*$-modules.)

$$m(\tilde{A})A\hat{x} = f \cdot A\hat{x} \quad \text{for some } f \in \mathbb{C}.$$

On the other hand,

$$\tilde{J}A\hat{x} = kA\hat{x} \quad (k : \text{valency of } \Gamma).$$

Therefore,

$$m(\tilde{A}) - \frac{f}{k}\tilde{J} = O,$$

and (34.45) holds.

$\square$

Chapter 35

$\dim {E_1}^* T {E_1}^* \leq 5$

Monday, April 26, 1993

Theorem 35.1. *Let $\Gamma = (X, E)$ be distance regular of diameter $D \geq 3$. Assume Γ is Q-polynomial with respect to $E_0, E_1, \dots, E_D$. Fix a vertex $x \in X$, and write $E_i^* \equiv E_i^*(x)$, $T \equiv T(x)$.*

(i) Up to isomorphism, there are at most four thin irreducible T-modules with endpoint 1.

(ii) Suppose Γ is thin with respect to x. Then

$$\dim E_1^* T E_1^* \leq 5.$$

Proof.

(ii) is immediate from (i) and part (iv) of Lemma 34.3.

(i)

Claim 1. $E_1^* M E_1^* = \mathrm{Span}(\tilde{J}, E_1^*, \tilde{A})$.

Proof of Claim 1.

$$E_1^* M E_1^* = \mathrm{Span}\{E_1^*, E_1^* A E_1^*, E_1^* A_2 E_1^*, E_1^* A_3 E_1^*, \dots\}.$$

But $E_1^* A_h E_h^* = O$ if $h > 2$ (by Lemma 16.1). So,

$$E_1^* M E_1^* = \mathrm{Span}\{E_1^*, E_1^* A E_1^*, E_1^* A_2 E_1^*\}.$$

Also,

$$\tilde{J} = E_1^* J E_1^* \tag{35.1}$$
$$= E_1^* \left(\sum_{h=0}^{D} A_h \right) E_1^* \tag{35.2}$$
$$= E_1^* + E_1^* A E_1^* + E_1^* A_2 E_1^*. \tag{35.3}$$

So,

$$E_1^* M E_1^* = \mathrm{Span}\{E_1^*, E_1^* A E_1^*, \tilde{J}\}.$$

We are done, since $\tilde{A} = E_1^* A E_1^*$.

Claim 2. $E_1^* M M^* M E_1^* = \mathrm{Span}(\tilde{J}, E_1^*, \tilde{A}, \tilde{A}^2)$.

Proof of Claim 2. $\supseteq$: Clear.

$\subseteq$: In Lemma 33.4 (i), we say

$$E_1^* T = E_1^* E_0 M^* + E_1^* M + E_1^* E_1 M^* + E^* E_1 E_1^* M + \cdots.$$

In fact, the proof of that lemma gives a sequence;

$$E_1^* M M^* = E_1^* E_0 M^* + E_1^* M + E_1^* E_1 M^*, \tag{35.4}$$
$$E_1^* M M^* M = E_1^* E_0 M^* + E_1^* M + E_1^* E_1 M^* + E^* E_1 E_1^* M, \tag{35.5}$$
$$E_1^* M M^* M M^* = E_1^* E_0 M^* + E_1^* M + E_1^* E_1 M^* + E^* E_1 E_1^* M + E^* E_1 E_1^* M M^*, \tag{35.6}$$
$$\vdots \tag{35.7}$$

Multiply (35.5) through on the right by E_1^* to get

$$E_1^* M M^* M E_1^* = E_1^* M E_1^* + E_1^* E_1 E_1^* M E_1^* = \mathrm{Span}\{\tilde{J}, E_1^*, \tilde{A}, \tilde{A}^2\},$$

since $\tilde{J}^2, \tilde{A}\tilde{J} = \tilde{J}\tilde{A} \in \mathrm{Span}\{\tilde{J}\}$.

This proves Claim 2.

Now, let W denote any irreducible T-module with endpoint 1, and pick $0 \neq v \in E_1^* W$. Set

$$v_i^+ = E_i^* A_{i-1} E_1^* v, \quad v_i^- = E_i^* A_{i+1} E_1^* v, \ i \in \{1, \ldots, D\}.$$

We know by Lemma 34.2 (ii) that W is thin if and only if v_i^+, v_i^- are linearly dependent for all $i \in \{2, \ldots, D-1\}$.

In general,

$$\Phi_i = \det \begin{pmatrix} \|v_i^+\|^2 & \langle v_i^+, v_i^- \rangle \\ \langle v_i^+, v_i^- \rangle & \|v_i^-\|^2 \end{pmatrix} \geq 0$$

with equality if and only if v_i^+, v_i^- are linearly dependent, (because Φ_i is the determinant of a Gram matrix).

Let i be an integer in $\{2, \dots, D-1\}$.

Claim 3. There exists $p^{++} \in \mathrm{C}[\lambda]$, $\deg p^{++} \leq 2$ (that depends only on the intersection numbers) such that

$$\|v_i^+\|^2 = \|v\|^2 p^{++}(a_0(W)).$$

Proof of Claim 3.

$$\|v_i^+\|^2 = \bar{v}^\top E_1^* A_{i-1} E_i^* E_i^* A_{i-1} E_1^* v = \bar{v}^\top E_1^* A_{i-1} E_i^* A_{i-1} E_1^* v.$$

But,

$$E_1^* A_{i-1} E_i^* A_{i-1} E_1^* \in E_1^* M M^* M E_1^* = \mathrm{Span}(\tilde{J}, E_1^*, \tilde{A}, \tilde{A}^2)$$

by Claim 2.

So, there exists $\alpha \in \mathbb{C}$, and $p^{++} \in \mathbb{C}[\lambda]$ with $\deg p^{++} \leq 2$ such that

$$E_1^* A_{i-1} E_i^* A_{i-1} E_1^* = \alpha \tilde{J} + p^{++}(\tilde{A}), \quad (\tilde{A}^0 = E_1^*).$$

Now,

$$\|v_i^+\|^2 = \bar{v}^\top (\alpha \tilde{J} + p^{++}(\tilde{A})) v = \|v\|^2 p^{++}(a_0(W)),$$

since $\tilde{J}v = 0$, and $\tilde{A}v = a_0(W)v$.

This proves Claim 3.

Similarly, there exist $p^{--}, p^{+-} \in \mathbb{C}[\lambda]$ with $\deg p^{--}, \deg p^{+-} \leq 2$ such that

$$\|v_i^-\|^2 = \|v\|^2 p^{--} p(a_0(W)),\ \langle v_i^+, v_i^- \rangle = \|v\|^2 p^{+-}(a_0(W)).$$

Claim 4. $E_1^* A_{i-1} E_i^* A_{i+1} E_1^* = (\tilde{J} - \tilde{A} - E_1^*) p^2_{i-1,i+1}$. In particular,

$$p^{+-}(\lambda) = -p^2_{i-1,i+1}(\lambda + 1).$$

Proof of Claim 4. Pick vertices $y, z \in X$ such that $\partial(x, y) = \partial(x, z) = 1$.

$$\begin{aligned}
(\mathrm{LHS})_{yz} &= \sum_{w \in X} (E_1^* A_{i-1} E_i^*)_{yw} (E_i^* A_{i+1} E_1^*)_{wz} && (35.8)\\
&= \sum_{w \in X, \partial(y,w) = i-1, \partial(x,w) = i, \partial(w,z) = i+1} 1 && (35.9)\\
&= \begin{cases} 0 & \text{if } \partial(y,z) = 0, \\ 0 & \text{if } \partial(y,z) = 1, \\ p^2_{i-1,i+1} & \text{if } \partial(y,z) = 2, \end{cases} && (35.10)\\
&= \mathrm{RHS}_{yz}. && (35.11)
\end{aligned}$$

Note that $E_1^* A_2 E_1^* = \tilde{J} - \tilde{A} - E_1^*$.

Now,

$$\begin{aligned} \langle v_i^+, v_i^- \rangle &= \bar{v}^\top E_1^* A_{i-1} E_i^* A_{i+1} E_1^* v && (35.12)\\ &= p^2_{i-1,i+1}(\bar{v}^\top(\tilde{J} - \tilde{A} - E_1^*)v) && (35.13)\\ &= -(a_0(W)+1)p^2_{i-1,i+1}\|v\|^2. && (35.14) \end{aligned}$$

Claim 5. $\deg p^{++} = \deg p^{--} = 2$. (only need for some i)

Proof of Claim 5. We need to calculate p^{++}, p^{--}.

HS MEMO

Pick vertices $y, z \in X$ such that $\partial(x,y) = \partial(x,z) = 1$. Then

$$(E_1^* A_{i-1} E_i^* A_{i-1} E_1^*)_{yz} = |\Gamma_{i-1}(y) \cap \Gamma_i(x) \cap \Gamma_{i-1}(z)|,$$

which is equal to $p^1_{i-1,i}$ if $\partial(y,z) = 0$.

$$(E_1^* A_{i+1} E_i^* A_{i+1} E_1^*)_{yz} = |\Gamma_{i+1}(y) \cap \Gamma_i(x) \cap \Gamma_{i+1}(z)|,$$

which is equal to $p^1_{i+1,i}$ if $\partial(y,z) = 0$.

Conclusion.

$$\begin{aligned} \Phi_i &= \det \begin{pmatrix} \|v_i^+\|^2 & \langle v_i^+, v_i^- \rangle \\ \langle v_i^+, v_i^- \rangle & \|v_i^-\|^2 \end{pmatrix} \geq 0 && (35.15)\\ &= \|v\|^4 (p^{++}(\lambda) p^{--}(\lambda) - (p^2_{i-1,i+1})^2 (\lambda+1)^2 && (35.16)\\ &\geq 0, && (35.17) \end{aligned}$$

where $\lambda = a_0(W)$.

W is thin if and only if $\Phi_i(\lambda) = 0$ for all $i \in \{2, \dots, D-1\}$.

Each Φ_i is degree 4 solutions for λ. Since λ determines the isomorphism class of W by Lemma 34.3 (iii), there are at most 4 different thin irreducible modules W of endpoint 1 up to isomorphism.

□

Note. In fact $\Phi_i(\lambda)$ is independent of i up to scalar multiple for $i \in \{2, \dots, D-1\}$.

If Γ has classical parameters (q, D, α, β), the roots are;

$$\beta - \alpha - 1, -1, -q-1, dq\frac{q^{D-1}-1}{q-1} - 1.$$

Chapter 36

Dual Endpoint

Wednesday, April 28, 1993

Let $\Gamma = (X, E)$ be distance regular of diameter $D \geq 3$, Q-polynomial with respect to $E_0, E_1, \dots, E_D$. Fix a vertex $x \in X$, write $E_i^* \equiv E_i^*(x)$, $T \equiv T(x)$.

Let W be an irreducible T-module of diameter d.

Recall that the endpoint

$$r(W) = \min\{i \mid 0 \leq i \leq D, E_i^* W \neq 0\}.$$

Definition 36.1. The dual endpoint (with respect to above ordering $E_0, E_1, \dots, E_D$),

$$r^*(W) = \min\{i \mid 0 \leq i \leq D, E_i W \neq 0\}.$$

$$r(W) = 0 \leftrightarrow r^*(W) = 0 \leftrightarrow W: \text{ trivial } T\text{-module},$$

(by Lemma 10.1).

Suppose W is thin. Then W is dual thin. (See Corollary 9.1.)

Moreover, $\{i \mid E_i W \neq 0\}$ is a subinterval of $\{0, 1, \dots, D\}$. (same proof as for distance regular)

HS MEMO

Dual version of Lemma 4.1.

Lemma 4.1'. Let $A^* \equiv A_1^*(x)$, W an irreducible T-moduoe, and $d^* = \{i \mid E_i W \neq 0\}| - 1$.

(*i*) $E_i A^* E_j = 0$ if $|i - j| > 1$, $E_i A^* E_j \neq 0$ if $|i - j| = 1$, $\quad 0 \leq i, j \leq d^*(x)$.
(*ii*) $A^* E_j W \subseteq E_{j-1} W + E_j W + E_{j+1} W$, $0 \leq j \leq d^*(x)$. ($E_i W = 0$ if $i < j$ or $i > d^*(x)$.)

(iii) $E_jW \neq 0$ if $r^* \le j \le r^* + d^*$, $E_jW = 0$ if $0 \le j \le r^*$ or $r^* + d^* < j \le d^*(x)$.
(iv) $E_iA^*E_jW \neq 0$, if $|i-j| = 1$ $(r^* \le i, j \le r^* + d^*)$.

Proof of 4.1'

(i) By Lemma 20.3,

$$E_iA^*E_j = O \leftrightarrow q^j_{i1} = 0.$$

By Lemma 22.2,

$$\Gamma\text{: } Q\text{-polynomial} \leftrightarrow q^j_{i1} \begin{cases} = 0 & \text{if } |j-i| > 1, \\ \neq 0 & \text{if } |j-i| = 1. \end{cases} \tag{36.1}$$

$$\leftrightarrow E_iA^*E_j \begin{cases} = O & \text{if } |j-i| > 1, \\ \neq O & \text{if } |j-i| = 1. \end{cases} \tag{36.2}$$

(ii) We have

$$\begin{aligned} A^*E_jW &= \left(\sum_{i=0}^{D} E_i\right) A^*E_jW && (36.3)\\ &= E_{j-1}A^*E_jW + E_jA^*E_jW + E_{j+1}A^*E_jW && (36.4)\\ &\subseteq E_{j-1}W + E_jW + E_{j+1}W. && (36.5) \end{aligned}$$

(iii) Suppose $E_jW = 0$ for some $j \in \{r^*, \dots, r^* + d^*\}$. Then $r^* < j$ by the definition of r^*. Set

$$\widetilde{W} = E_{r^*}W + E_{r^*+1}W + \cdots + E_{j-1}W.$$

Observe $0 \subsetneq \widetilde{W} \subsetneq W$. Also $A\widetilde{W} \subseteq \widetilde{W}$ by (ii), and $E^*_i\widetilde{W} \subseteq \widetilde{W}$ for every i by construction.

Thus, $T\widetilde{W} \subseteq \widetilde{W}$, contradicting W being irreducible.

(iv) Suppose $E_{j+1}A^*E_jW = 0$ for some $j \in \{r^*, \dots, r^* + d^* - 1\}$. Then,

$$\widetilde{W} = E_{r^*}W + E_{r^*+1}W + \cdots + E_jW$$

is T-invariant. If $E_{j-1}A^*E_jW = 0$ for some $j \in \{r^* + 1, \dots, r^* + d^*\}$, then

$$\widetilde{W} = E_jW + E_{j+1}W + \cdots + E_{r^*+d^*}W$$

is T-invariant. Moreover, $0 \subsetneq \widetilde{W} \subsetneq W$ in both cases. A contradiction.

Definition. Let W be an irreducible dual thin T-module with dual endpoint r^* and diameter d^*.

Let $a_i^* = a_i^*(W) \in \mathbb{C}$ satisfying

$$E_{r^*+i}A^*E_{r^*+i}|_{E_{r^*+i}W} = a_i^* \cdot 1|_{E_{r^*+i}W}.$$

Let $x_i^* = x_i^*(W) \in \mathbb{C}$ satisfying

$$E_{r^*+i-1}A^*E_{r^*+i}A^*E_{r^*+i-1}|_{E_{r^*+i-1}W} = x_i^* \cdot 1||_{E_{r^*+i-1}W}.$$

Lemma 9.1'. With above notation, the following hold.

(i) $a_i^* \in \mathbb{R}$ for all $i \in \{0, \dots, d^*\}$.

(ii) $x_i^* \in \mathbb{R}^{>0}$ for all $i \in \{1, \dots, d^*\}$.

(iii) Pick $0 \neq w_0^* \in E_{r^*}^*W$. Set $w_i^* = E_{r^*+i}{A^*}^i w_0^*$ for all i. Then

$(iiia)$ $w_0^*, w_1^*, \dots, w_{d^*}^*$ is a basis for W, $w_{-1}^* = w_{d^*+1}^* = 0$.

$(iiib)$ $A^*w_i^* = w_{i+1}^* + a_i^*w_i + x_i^*w_{i-1}^*$ for all $i \in \{0, \dots, d^*\}$.

(iv) Define $p_0^*, p_1^*, \dots, p_{d^*+1}^* \in \mathbb{R}[\lambda]$ by

$$p_0^* = 1, \quad \lambda p_i^* = p_{i+1}^* + a_i^*p_i^* + x_i^*p_{i-1}^* \quad \text{for all } i \in \{0, \dots, d^*\}, \quad p_{-1}^* = 0.$$

(iva) $p_i^*(A^*)w_0^* = w_i^*$, for all $i \in \{0, \dots, d^*+1\}$.

(ivb) $p_{d^*+1}^*$ is the minimal polynomial of $A^*|_W$.

Proof of Lemma 9.1'

(i) Recall

$$A^* = \sum_{j=0}^{D} \theta_j^* E_j^*, \quad \theta_j^* = q_1(j) = |X|(E_1)_{xy} \in \mathbb{R},\ \partial(x, y) = j.$$

a_i^* is an eigenvalue of a real symmetric matrix $E_{r^*+i}A^*E_{r^*+i}$.

(ii) Let $B = E_{r^*+i}^*AE_{r^*+i-1}^*$.

Then, x_i^* is an eigenvalue of a real symmetrix matrix $B^\top B$. Let $\mathrm{Span}\{v_{i-1}\} = E_{r^*+i-1}W$, and $Bv_{i-1} \neq 0$ by Lemma 4.1' (iv) for $i \in \{1, \dots, d^*\}$. So, $x_i \in \mathbb{R}^{>0}$ for all $i \in \{1, \dots, d^*\}$.

$(iiia)$ Observe

$$w_i^* = E_{r^*+i}A^*E_{r^*+i-1}^*w_{i-1}^* \quad \text{for all } i \in \{1, \dots, d^*\}.$$

So $w_i^* \neq 0$ for all $i \in \{1, \dots, d^*\}$ by Lemma 4.1' (iv).

Hence,

$$W = \mathrm{Span}(w_0^*, \dots, w_d^*)$$

by Lemma 4.1' (iii).

$(iiib)$ We have that

$$\begin{aligned} A^* w_i^* &= E_{r^*+i+1} A^* w_i^* + E_{r^*+i} A^* w_i^* + E_{r^*+i-1} A^* w_i^* && (36.6)\\ &= w_{i+1}^* + E_{r^*+i} A^* E_{r^*+i} w_i^* + E_{r^*+i-1} A^* E_{r^*+i} A^* E_{r^*+i-1} w_{i-1} && (36.7)\\ &= w_{i+1}^* + a_i^* w_i^* + x_i^* w_{i-1}^*. && (36.8) \end{aligned}$$

(iva) Clear for $i = 0$. Assume it is valid for $0, \dots, i$.

$$p_{i+1}^*(A^*) w_0^* = (A^* - a_i^* I) w_i^* - x_i^* w_{i-1}^* = w_{i+1}^*.$$

(ivb) By definition,

$$p_{d^*+1}^*(A^*) w_0^* = 0.$$

Since $W = \{p(A^*) w_0^* \mid p \in \mathbb{C}[\lambda]\}$, $p_{d^*+1}^*(A^*) W = 0$, and $p_{d^*+1}^*$ is a minimal polynomial, as $w_0^*, w_1^*, \dots, w_{d^*}^*$ is a basis of W.

Corollary 9.1'. With the notation above, let W be a dualthin irreducible T-module with dual endpoint $r^*(W)$, and dual diameter d^*. Then,

(i) W is thin,

(ii) $d^* = d = |\{i \mid E_i^* W \neq 0\}| - 1$.

Proof of Corollary 9.1'

Set as in Lemma {4.1}'.

$$w_i^* = p_i^*(A^*) w_0^* \in E_{r^*+i} W.$$

Then, $w_0^*, w_1^*, \dots, w_{d^*}^*$ is a basis for W. We have $W = M^* w_0^*$.

So, $E_i^* W = E_i^* M^* w_0^* = \mathrm{Span}(E_i^* w_0^*)$.

Thus, W is thin, and so, we have (ii).

Suppose $r(W) = 1$. Then $d(W) = D - 2$ or $D - 1$ by Lemma 14.1 (iii). See also Lemma 14.2.

Case $d(W) = D - 2$. Then

$E_1 W = 0$ implies $r^*(W) = 2$.

$E_1 W \neq 0$ implies $r^*(W) = 1$.

Case $d(W) = D - 1$. Then

$r^*(W) = 1$.

Up to isomorphism,

there are at most 3 thin irreducible T-modules with $r(W) = 1$ and $r^*(W) = 1$,

there are at most 1 thin irreducible T-module with $r(W) = 1$ and $r^*(W) = 2$,

there are none thin irreducible T-modules with $r(W) = 1$ and $r^*(W) > 2$.

By dual argument,

there are at most 3 thin irreducible T-modules with $r^*(W) = 1$ and $r(W) = 1$,

there are at most 1 thin irreducible T-module with $r^*(W) = 1$ and $r(W) = 2$,

there are none thin irreducible T-modules with r and $r(W) > 2$.

Conjecture 36.1. *Let $\Gamma = (X, E)$ be a thin distance regular graph of diameter $D \geq 3$. Let E_1 be any primitive idempotent not equal to E_0.*

Then the following are equivalent.

(i) For every vertex $x \in X$, there is no irreducible T-module W with $r(W) > 2$, and $E_1 W \neq 0$, there exists at most 1 irreducible T-module with $r(W) = 2$, and $E_1 W \neq 0$, and there exist at most 3 irreducible T-modules W with $r(W) = 1$, and $E_1 W \neq 0$.

(ii) Γ is Q-polynomial with respect to E_1.

Conjecture 36.2. *Let $\Gamma = (X, E)$ be distance regular of diameter $D \geq 3$, Q-polynomial with respect to $E_0, E_1, \dots, E_D$. Fix a vertex $x \in X$, and write $E_i^* \equiv E_i^*(x)$, $T \equiv T(x)$. Let W denote an irreducible T-module with endpoint r, dual endpoint r^*, diameter d and dual diameter d^*.*

Then the following hold.

(i) $d = d^*$.

(ii) there exists $s \in \{r, \dots, r+d\}$ such that

$$1 = \dim E_r^* W \leq \dim E_{r+1}^* W \leq \cdots \leq \dim E_s^* W \geq \cdots \geq \dim E_{r+d}^* W.$$

(iii) there exists $s^* \in \{r^*, \dots, r^*+d^*\}$ such that

$$1 = \dim E_{r^*} W \leq \dim E_{r^*+1} W \leq \cdots \leq \dim E_{s^*} W \geq \cdots \geq \dim E_{r^*+d^*} W.$$

Let $\Gamma = (X, E)$ be distance regular of diameter $D \geq 3$, Q-polynomial with respect to $E_0, E_1, \dots, E_D$. Fix a vertex $x \in X$, write $E_i^* \equiv E_i^*(x)$ and $T \equiv T(x)$. Let W denote an irreducible module with endpoint 1.

Conjecture 36.3. *The following are equivalent.*

(i) The sequence $\dim E_1^* W, \dim E_2^* W, \ldots, E_D^* W$ *equals*

$$1, 2, 2, \ldots, 2, 1.$$

(ii) $v, Av, A_2 v, \ldots, A_{D-2} v,\ v^*, A^* v^*, A_2^* v^*, \ldots, A_{D-2}^* v^*$ *is a basis for* W*, where*

$$0 \neq v \in E_i^* W, \text{ and } v^* = |X| E_1 v.$$

(iv) $v_1^+, v_2^+, \ldots, v_D^+, v_2^-, v_3^-, \ldots, v_{D-1}^-$ *is a basis for* W*, where*

$$v_i^+ = E_i^* A_{i-1} v, \quad v_i^- = E_i^* A_{i+1} v.$$

Problem. Let B denote the orthogonal basis for W obtained by applying the Gram-Schemidt procedure to be basis in *(iv)*.

Find the matrix representation A with respect to this basis.

I believe the entries are necely foctorable expressions in the basic variables,

$$q, s, s^*, r_1, r_1.$$

(Hint: use Theorem 35.1.)

If not, find some nice basis for W, and find the matrices representing A, A^* with respect to this basis.

Perhaps, some orthogonal basis based on *(iii)*.

Algebraically, everything is determined by the intersection numbers and $a_0(W)$.

Combinatorically, certain quantities mulst be nonnegative integers. Does this give some new bounds, or other information on $a_0(W)$?

Chapter 37

Generalized Adjacency Matrix

Friday, April 30, 1993

Lemma 37.1. *Let $\Gamma = (X, E)$ be a distance-regular graph of diameter $D \geq 3$, and Q-polynomial with respect to $E_0, E_1, \dots, E_D$. Fix a vertex $x \in X$, and write $E_i^* \equiv E_i^*(x)$, and $T \equiv T(x)$. Let W be an irreducible T-module of endpoint 1. If $\dim E_2^* W = 1$, then W is thin.*

Proof. Pick $0 \neq v \in E_1^* W$.

We want to show that

- $FR^i v \in \operatorname{Span}(R^i v)$ for $i \in \{0, \dots, D-1\}$.
- $LR^i v \in \operatorname{Span}(R^{i-1} v)$ for $i \in \{1, \dots, D-1\}$.

We have that

(1) $FR^2 E_j^* \in \operatorname{Span}(RFRE_j^*, R^2 FE_j^*, R^2 E_j^*)$ for $i \in \{0, \dots, D-3\}$.

(2) $LR^2 E_j^* \in \operatorname{Span}(RLRE_j^*, R^2 LE_j^*, F^2 RE_j^*, FRFE_j^*, RF^2 E_j^*, RFE_j^*, FRE_j^*, RE_j^*)$ for $i \in \{0, \dots, D-3\}$

by Corollary 30.1.

Claim (a) $FR^i v \in \operatorname{Span}(R^i v)$ for $i \in \{0, \dots, D-2\}$,
(b) $LR^i v \in \operatorname{Span}(R^{i-1} v)$ for $i \in \{1, \dots, D-2\}$.

HS MEMO

Proof of Claim.

(a) By Lemma 34.2, and our assumption

$$\dim E_1^* W = \dim E_2^* W = 1.$$

So, $Rv \neq 0$, and $E_2^* W = \mathrm{Span}(Rv)$.

We may assume $i \geq 2$. Then $R^{i-2}v \in E_{i-1}^* W$,

$$\begin{aligned} FR^i v &= FR^2 R^{i-2} v, \quad \text{if } \ i \leq D-2, && (37.1)\\ &= R(FR + RF + R)R^{i-2}v && (37.2)\\ &\in R(\mathrm{Span}(R^{i-1}v)) && (37.3)\\ &= \mathrm{Span}(R^i v), && (37.4)\end{aligned}$$

by the induction hypothesis.

(b) If $i \leq D-2$, then $R^{i-2}v \in E_{i-1}^* W$ with $i-1 \leq D-3$. Hence,

$$\begin{aligned} LR^i v &= LR^2(R^{i-2}v) && (37.5)\\ &= (RLR + R^2L + F^2R + FRF + RF^2 + RF + FR + R)R^{i-2}v && (37.6)\\ &\in \mathrm{Span}(R^{i-1}v), && (37.7)\end{aligned}$$

by induction and (a).

Suppose $R^{D-1}v = 0$. Then,

$$\mathrm{Span}(v, Rv, \dots, R^{D-2}v) = \widetilde{W}$$

is invariant under M and M^*, hence, under T.

Since W is irreducible, $W = \widetilde{W}$, and W is thin in this case.

Suppose $R^{D-1}v \neq 0$.

Observe: $v, Av, \dots, A^{D-1}v \in \mathrm{Span}(v, Rv, \dots, R^{D-1}v)$.

Hence, each $R^i v$ is a polynomial of degree i in A applied to v, and

$$\mathrm{Span}(v, Av, \dots, A^{D-1}v) = \mathrm{Span}(v, Rv, \dots, R^{D-1}v) = \mathrm{Span}(v, A_1v, \dots, A_{D-1}v).$$

Also,

$$A_D v = Jv - \left(\sum_{h=0}^{D-1} A_h\right) v \in \mathrm{Span}(v, A_1 v, \dots, A_{D-1}v).$$

Thus,

$$Mv = \mathrm{Span}(v, Rv, \dots, R^{D-1}v).$$

Therefore,

$$\mathrm{Span}(v, Rv, \dots, R^{D-1}v) = \widetilde{W}$$

is invariant under M, M^*, and hence T. We have $W = \widetilde{W}$ and W is thin. $\square$

Definition 37.1. Let $\Gamma = (X, E)$ be any regular graph (not necessarily connected).

Let A be the adjacency matrix of Γ, and let J be the all 1's matrix.

Pick $O \neq B \in \mathrm{Mat}_X(\mathbb{C})$.

B is a generalized adjacency matrix , if

(i) for all vertices $x, y \in X$, $B_{xy} \neq 0$ implies $A_{xy} \neq 0$ or $x = y$,

(ii) B is in the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by A and J.

Example 37.1. Any nonzero matrix of form

$$\alpha A + \beta I \quad (\alpha, \beta \in \mathbb{C})$$

is a generalized adjacency matrix.

If Γ is distance regular, all generalized adjacecy matrices are of this form.

Let $\Gamma = (X, E)$ be a distance-regular graph of diameter $D \geq 3$. Assume Γ is thin, and Q-polynomial.

Pick a vertex $x \in X$, and write $E_i^* \equiv E_i^*(x)$, $T \equiv T(x)$. Then,

$$E_1^* T E_1^* = \mathrm{Span}(\tilde{J}, E_1^*, \tilde{A}, \tilde{A}^2, \tilde{A}^3),$$

and $\dim E_1^* T E_1^* \leq 5$.

We will produce a 'nice' spanning set

$$E_1^* T E_1^* = \mathrm{Span}(\tilde{J}, E_1^*, \tilde{A}, A^+ \; (= R^{-1} E_2^* A E_1^*), A^+ \tilde{A}).$$

Lemma 37.2. *Let $\Gamma = (X, E)$ be a thin distance-regular graph of diameter $D \geq 4$.*

Fix a vertex $x \in X$, and write $E_i^* \equiv E_i^*(x)$ and $R \equiv R(x)$.

Let Γ_1 denote the vertex subgraph induced on the first subconstituent of Γ relative to x. Then,

$$\Delta = (R^{-1})^{i-1} E_i^* A_i E_1^*$$

is a generalized adjacency matrix for Γ_1 for all $i \in \{1, \ldots, D-3\}$.

Proof. Write $T \equiv T(x)$. Fix $i \in \{1, \ldots, D-3\}$.

Recall $R^{-1} \in T$ by Lemma 31.1 (iv).

Since $E_{i-1}^* R^{-1} E_i^* = R^{-1} E_i^*$ by Lemma 31.1 (ii),

$$\Delta \in E_1^* T E_1^* = \mathrm{Span}(\tilde{J}, E_1^*, \tilde{A}, \tilde{A}^2, \ldots)$$

by Lemma 34.3 (iv).

Hence, Δ satisfied the condition (ii) of Definition 37.1.

To show (i), pick vertices $y, z \in X$ such that

$$\partial(x,y) = \partial(x,z) = 1, \quad \partial(y,z) = 2.$$

We need to show

$$\Delta_{yz} = 0.$$

Suppose $\Delta_{yz} \neq 0$. Then,

$$\langle \Delta\hat{y}, \hat{z} \rangle \neq 0.$$

We will show this cannot occur.

Notation: Set

$$E^*_{ij} = E^*_i(x)E^*_j(y),\ i,j \in \{0,1,\dots,D\}.$$

Then,

$$E^*_{ij}V = \mathrm{Span}(\hat{w} \mid w \in X, \partial(x,w) = i, \partial(y,w) = j) \text{ for } i,j \in \{0,1,\dots,D\}.$$

Let δ denote the all 1's vector in V. Let

$$\delta_{ij} = E^*_{ij}\delta = \sum_{w\in X, \partial(x,w)=i, \partial(y,w)=j} \hat{w}.$$

Now,

$$\Delta\hat{y} \in E^*_1(x)V = E^*_{10}V + E^*_{11}V + E^*_{12}V \quad \text{(orthogonal direct sum)}.$$

So, there exist $\delta^+_{10} \in E^*_{10}V$, $\delta^+_{11} \in E^*_{11}V$, and $\delta^+_{12} \in E^*_{12}V$ such that

$$\Delta\hat{y} = \delta^+_{10} + \delta^+_{11} + \delta^+_{12}.$$

Observe: $\hat{z} \in E^*_{12}V$ is not orthogonal to $\Delta\hat{y}$.

So, $\delta^+_{12} \neq 0$.

Observe:

$$\begin{aligned}
R^{i-1}(\delta^+_{10} + \delta^+_{11} + \delta^+_{12}) &= R^{i-1}\Delta\hat{y} &(37.8)\\
&= R^{i-1}(R^{-1})^{i-1}E^*_iA_iE^*_1\hat{y} &(37.9)\\
&= E^*_iA_iE^*_1\hat{y} &(37.10)\\
&= \delta_{ii} &(37.11)\\
&\in E_{ii}V. &(37.12)
\end{aligned}$$

HS MEMO

It is because on each irreducible thin module with standard basis $w_r, w_{r+1}, \dots, w_{r+d}$,

$$R^{-1}w_i = w_{i-1},\ i > r,\ R^{-1}w_r = 0,$$

and E_1^*V is an orthogonal direct sum of irreducible modules and $r \leq 1$.

But we can control $R^{i-1}\delta_{10}^+$, $R^{i-1}\delta_{11}^+$, also.

Claim. $RE_{jj}^*V \subseteq E_{j+1,j+1}^*V + E_{j+1,j}^*V,\ j \in \{1, \dots, D-1\}$.

Proof of Claim. Clear.

By Claim

$$R^{i-1}\delta_{10}^+ \in E_{i,i-1}^*V, \quad \text{and} \tag{37.13}$$

$$R^{i-1}\delta_{11}^+ \in E_{i,i-1}^*V + E_{i,i}^*V. \tag{37.14}$$

Hence, we conclude that

$$R^{i-1}\delta_{12}^+ = R^{i-1}\Delta\hat{y} - R^{i-1}\delta_{10}^+ - R^{i-1}\delta_{11}^+ \in E_{i,i-1}^*V + E_{ii}^*V.$$

But now

$$0 = E_{i,i+1}^*R^{i-1}\delta_{12}^+ = E_{i,i+1}^*A^{i-1}E_{12}^*\delta_{12}^+ = R(y)^{i-1}\delta_{12}^+. \tag{37.15}$$

By Lemma 32.1 (ii),

$$R(y)^{i-1} : E_2^*(y)V \longrightarrow E_{i+1}^*V$$

is one-to-one, since Γ is thin, and $i-1 \leq D-4$.

So, $\delta_{12}^+ = 0$ by (37.15).

But this contradicts (2). Hence our assumption $\Delta_{yz} \neq 0$ is false, and the condition (i) of the definition of generalised adjacency matrices is satisfied.

This proves the lemma. $\square$

Chapter 38

An Injection from E_{11}^* to E_{22}^*

Monday, May 3, 1993

Lemma 38.1. *Let $\Gamma = (X, E)$ be a thin distance-regular graph of diameter $D \geq 5$, and Q-polynomial with respect to $E_0, E_1, \ldots, E_D$. Pick vertices $x, y \in X$ such that $\partial(x, y) = 1$, and write $E^*_{ij} := E^*_i(x)E^*_j(y)$ for $i, j \in \{0, 1, \ldots, D\}$. Then the following hold.*

(i) $E^*_{22}AE^*_{11} : E^*_{11}V \to E^*_{22}V$ is one-to-one.

(ii) For every $z \in X$ such that $\partial(x, z) = \partial(y, z) = 1$, there is $w \in X$ such that

$$\partial(w, x) = \partial(w, y) = 2, \ \partial(w, z) = 1.$$

Proof.

(i) Write $E^*_i \equiv E^*_i(x)$, $R \equiv R(x)$, $F \equiv F(x)$, $L \equiv L(x)$, and $T \equiv T(x)$.

Suppose there exists

$$0 \neq v \in E^*_{11}V \ \text{ such that } \ E^*_{22}AE^*_{11}v = 0. \tag{38.1}$$

Claim 1. $E^*_{34}A^2E^*_{12}AE^*_{11}v \neq 0$.

Proof of Claim 1. Recall by Lemma 32.1 (ii), $(3 \leq 5 - 2 \leq D - 2t)$,

$$R(y)^3 : E^*_1(y)V \to E^*_4(y)V$$

is one-to-one.

Since $v \in E_1^*(y)V$, we find

$$\begin{align} 0 &\neq R^3(y)v \tag{38.2}\\ &= E_4^*(y)A^3E_1^*(y)v \tag{38.3}\\ &= E_4^*(y)A^2E_2^*(y)AE_{11}^*v \tag{38.4}\\ &= E_4^*(y)A^2\left(\sum_{h=0}^{D} E_{h,2}^*\right)AE_{11}^*v \tag{38.5}\\ &= E_4^*(y)A^2(E_{12}^*+E_{22}^*)AE_{11}^*v \tag{38.6}\\ &= E_4^*(y)A^2E_{12}^*AE_{11}^*v \tag{38.7}\\ &= E_{34}^*(y)A^2E_{12}^*AE_{11}^*v, \tag{38.8} \end{align}$$

by (38.1). This proves the claim.

By Theorem 30.1 (i),

$$0 = (g_3^-R^2F + RFR + g_3^+FR^2 - \gamma R^2)E_1^*. \tag{38.9}$$

HS MEMO

Theorem 30.1 (i) states

$$(g_i^-FL^2 + LFL + g_i^+L^2F - \gamma L^2)E_i^* = O \quad \text{for } i \in \{2, \dots, D\}.$$

For $i = 3$,

$$E_1^*(g_3^-FL^2 + LFL + g_3^+L^2F - \gamma L^2)E_3^* = O.$$

Taking the transpose, we have

$$E_3^*(g_3^-R^2F + RFR + g_3^+FR^2 - \gamma R^2)E_1^* = O.$$

Hence, we have (38.9).

Multiplying each term on the left by $E_4^*(y)$, on the right by $E_1^*(y)$, we find

$$\begin{align} O &= g_3^-E_{34}^*R^2FE_{11}^* + E_{34}^*RFRE_{11}^* + g_3^+E_{34}^*FR^2E_{11}^* - \gamma E_{34}^*R^2E_{11}^* \tag{38.10}\\ &= g_3^-E_{34}^*A^2E_{12}^*AE_{11}^* + E_{34}^*AE_{23}^*AE_{22}^*AE_{11}^* + g_3^+E_{34}^*AE_{33}^*AE_{22}^*AE_{11}^*. \tag{38.11} \end{align}$$

Applying this to v, we find by (38.1) that

$$0 = g_3^-E_{34}^*A^2E_{12}^*AE_{11}^*v.$$

So, $g_3^- = 0$ by Claim 1. But by Lemma 30.1,

$$g_3^- = \frac{\theta_1^* - \theta_0^*}{\theta_1^* - \theta_3^*} \neq 0,$$

a contradiction.

□

Let Γ, x, y be as in Lemma 38.1. We saw in Lemma 37.2,

$$R^{-1}E_2^*A_2E_1^*\hat{y} = \delta_{10}^+ + \delta_{11}^+,$$

where

$$\delta_{10}^+ \in E_{10}^*V = \mathrm{Span}(\hat{y}), \quad \delta_{11}^+ \in E_{11}^*V.$$

Definition 38.1. Define $\Psi = \Psi(x, y) \in \mathbb{C}$ by $\delta_{10}^+ = \Psi\hat{y}$.

We will show that $\Psi(x, y)$ is independent of x, y.

Observe $R^{-1}, A_i, E_i^* \in \mathrm{Mat}_X(\mathbb{Q})$. So $\Psi \in \mathbb{Q}$.

Firstly, show

$$\Psi(x, y) = \Psi(y, x).$$

Lemma 38.2. *With the notation of Lemma 38.1, the following hold.*

(i) $E_{22}^*AE_{11}^*\delta_{11}^+ = \delta_{22}$.

(ii) $E_{21}^*AE_{11}^*\delta_{11}^+ = -\Psi(x, y)\delta_{21}$.

(iii) $\langle\delta_{11}^+, \delta_{11}\rangle = \frac{a_2}{c_2} - \Psi(x, y)$.

(iv) $\Psi(x, y) = \Psi(y, x)$.

(v) $E_{12}^*AE_{11}^*\delta_{11}^+ = -\Psi(x, y)\delta_{12}$.

Proof. Write $\Psi \equiv \Psi(x, y)$, $R \equiv R(x)$, $E_i^* \equiv E_i^*(x)$, etc.

(i) We have

$$\begin{aligned} R(\delta_{11}^+ + \Psi\hat{y}) &= R(\delta_{11}^+ + \delta_{10}^+) && (38.12)\\ &= R(R^{-1}(E_2^*A_2E_1^*))\hat{y} && (38.13)\\ &= E_2^*A_2E_1^*\hat{y} && (38.14)\\ &= \delta_{22}. && (38.15) \end{aligned}$$

So,

$$\begin{aligned} \delta_{22} &= R(\delta_{11}^+ + \Psi\hat{y}) && (38.16)\\ &= E_2^*AE_1^*(\delta_{11}^+ + \Psi\hat{y}) && (38.17)\\ &= E_{22}^*AE_{11}^*\delta_{11}^+ + \Psi E_{22}^*AE_{10}^*\hat{y}. && (38.18) \end{aligned}$$

The second term is zero.

(ii) We have

$$
\begin{aligned}
0 &= E^*_{21}\delta_{22} && (38.19)\\
&= E^*_{21}R(\delta^+_{11} + \Psi\hat{y}) && (38.20)\\
&= E^*_{21}AE^*_{11}\delta^+_{11} + \Psi E^*_{21}AE^*_{10}\hat{y} && (38.21)\\
&= E^*_{21}AE^*_{11} + \Psi\delta_{21}. && (38.22)
\end{aligned}
$$

(iii) We have

$$
\begin{aligned}
p^1_{22} &= \|\delta_{22}\|^2 && (38.23)\\
&= \langle \delta_{22}, \delta_{21} + \delta_{22} + \delta_{23}\rangle && (38.24)\\
&= \langle R(\delta^+_{11} + \Psi\hat{y}), \delta_{21} + \delta_{22} + \delta_{23}\rangle && (38.25)\\
&= \langle \delta^+_{11} + \Psi\hat{y}, L(\delta_{21} + \delta_{22} + \delta_{23})\rangle && (38.26)\\
&= b_1\langle \delta^+_{11} + \Psi\hat{y}, \delta_{10} + \delta_{11} + \delta_{12}\rangle && (38.27)\\
&= b_1(\langle \delta^+_{11}, \delta_{11}\rangle + \Psi). && (38.28)
\end{aligned}
$$

So,

$$\langle \delta^+_{11}, \delta_{11}\rangle = b_1^{-1}p^1_{22} - \Psi = \frac{a_2}{c_2} - \Psi.$$

HS MEMO

$$b_1^{-1}p^1_{22} = b_1^{-1}\frac{k_1}{k_1}p^1_{22} = b_1^{-1}\frac{1}{k_1}k_2p^2_{12} = b_1^{-1}\frac{b_1}{c_2}a_2 = \frac{a_2}{c_2}.$$

(iv) Interchanging roles of x, y above, we find there exists $\delta^{+'}_{11} \in E^*_{11}V$ such that

$$R(y)^{-1}E^*_2(y)A_2E^*_1(y)\hat{x} = \delta^{+'}_{11} + \Psi(y, x)\hat{y}.$$

Then,

$$E^*_{22}AE^*_{11}(\delta^{+'}_{11}) = \delta_{22}.$$

So,

$$E^*_{22}AE^*_{11}(\delta^+_{11} - \delta^{+'}_{11}) = 0.$$

Hence, $\delta^+_{11} = \delta^{+'}_{11}$ since

$$E^*_{22}AE^*_{11} : E^*_{11}V \to E^*_{22}V$$

is one-to-one.

Now,

$$\frac{a_2}{c_2} - \Psi(x, y) = \langle \delta^+_{11}, \delta_{11}\rangle = \langle \delta^{+'}_{11}, \delta_{11}\rangle = \frac{a_2}{c_2} - \Psi(y, x).$$

Thus,

$$\Psi(x, y) = \Psi(y, x).$$

(v) Immediate from (ii), (iv).

$\square$

Chapter 39

A^+ and A^-

Wednesday, May 5, 1993

Assume $\Gamma = (X, E)$ is thin, distance regular of diameter $D \geq 5$, and Q-polynomial with respect to $E_0, E_1, \ldots, E_D$.

Fix a vertex $x \in X$, write $E_i^* \equiv E_i^*(x)$, $R \equiv R(x)$, $T \equiv T(x)$.

Pick $y \in X$ with $\partial(x,y) = 1$. Write $E_{i,j}^* \equiv E^*(x)E^*(y)$, $\delta_{ij} = E_{ij}^*\delta$, and $\tilde{A} = E_1^* A E_1^*$.

Recall that $\delta_{11}^+ \in E_{11}^* V$ and

$$R^{-1} E_2^* A_2 E_1^* \hat{y} = \delta_{11}^+ + \Psi(x,y)\hat{y}.$$

We saw $\Psi(x,y) = \Psi(y,x)$. We shall show below that $\Psi(x,y)$ is independent of edge xy.

Lemma 39.1. *With the above notation, set $\Psi := \Psi(x,y)$. Then the following hold.*

(i) $\delta_{11}^- = \tilde{A}\delta_{11}^+ - \left(\frac{a_2}{c_2} - \Psi\right)\hat{y} + \Psi\delta_{12} \in E_{11}^* V$.

(ii) $\delta_{11}^-(x,y) = \delta_{11}^{-1}(y,x)$.

Proof.

(i) $\delta_{12}^- \in E_{12}^* V$, $\delta_{11}^- \in E_{11}^* V$ and $\delta_{10}^- \in E_{10}^* V$, and

$$\tilde{A}\delta_{11}^+ = \delta_{12}^- + \delta_{11}^- + \delta_{10}^-, \tag{39.1}$$

$$\delta_{12}^- = E_{12}^* A E_{11}^* \delta_{11}^+ = -\Psi(x,y)\delta_{12}, \tag{39.2}$$

by Lemma 38.2 (v).

Also, $\delta_{10}^- = \sigma\hat{y}$ for some $\sigma \in \mathbb{C}$, where

$$\sigma = \langle \tilde{A}\delta_{11}^+, \hat{y}\rangle = \langle \delta_{11}^+, \tilde{A}\hat{y}\rangle = \langle \delta_{11}^+, \delta_{11}\rangle = \frac{a_2}{c_2} - \Psi. \tag{39.3}$$

Solving for δ_{11}^- in (39.1), using (39.2) and (39.3), we have

$$\begin{aligned} \delta_{11}^- &= \tilde{A}\delta_{11}^+ - \delta_{12}^- - \delta_{10}^- && (39.4)\\ &= A\delta_{11}^+ + \Psi\delta_{12} - \left(\frac{a_2}{c_2} - \Psi\right)\hat{y}. && (39.5) \end{aligned}$$

(ii) Since

$$\delta_{11}^- = E_{11}^* A E_{11}^* \delta_{11}^+,$$

we have $\delta_{11}^+(x,y) = \delta_{11}^+(y,x)$.

□

Lemma 39.2. *With the above noation, $\Psi = \Psi(u,v)$ is independent of u, v, where $u, v \in X$, with $\partial(u,v) = 1$.*

Proof. Let x, y be as above $(x \sim y)$, and pick $z \in X$ such that $\partial(x,z) = 1$, but $z \neq y$. Then it suffices to show:

$$\Psi(x,y) = \Psi(x,z).$$

Case: $\partial(y,z) = 2$.

Set $\Delta := \tilde{A}R^{-1}E_2^* A_2 E_1^*$.

Observe: $\Delta \in E_1^* T E_1^*$ and $E_1^* T E_1^*$ is symmetrix by Lemma 33.4.

Hence, $\Delta_{yz} = \Delta_{zy}$.

Since $\Delta \in \mathrm{Mat}_X(\mathbb{R})$,

$$\langle \Delta\hat{y}, \hat{z}\rangle = \langle \Delta\hat{z}, \hat{y}\rangle.$$

But,

$$\begin{aligned} \langle \Delta\hat{y}, \hat{z}\rangle &= \langle \tilde{A}\delta_{11}^+ + \Psi(x,y)\hat{y}, \hat{z}\rangle && (39.6)\\ &= \langle \tilde{A}\delta_{11}^+, \hat{z}\rangle && (39.7)\\ &= \left\langle \delta_{11}^- + \left(\frac{a_2}{c_2} - \Psi\right)\hat{y} - \Psi(x,y)\delta_{12}, \hat{z}\right\rangle && (39.8)\\ &= -\Psi(x,y). && (39.9) \end{aligned}$$

Note that $\partial(x,y) = 2$ by Lemma 39.1 (i).

Similarly,

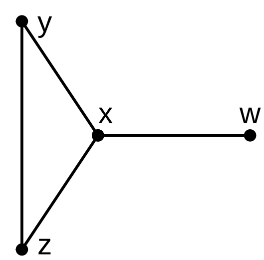
$$\langle \Delta \hat{z}, \hat{y} \rangle = -\Psi(x, z).$$

Hence, $\Psi(x, y) = \Psi(x, z)$.

Case: $\partial(y, z) = 1$.

By Lemma 38.1 (*ii*), there exists $w \in X$ such that

$$\partial(x, z) = 1,\ \partial(w, y) = 2,\ \partial(w, z) = 2.$$



Now,

$$\Psi(x, y) = \Psi(x, w) = \Psi(x, z)$$

from the first case. □

Lemma 39.3. *With the above notation, the following hold.*

(*i*) $A^+ := R^{-1} E_2^* A_2 E_1^* - \Psi E_1^*$, *and*

(*ii*) $A^- = \tilde{A} A^+ - \left(\frac{a_2}{c_2} - \Psi\right) E_1^* + \Psi(\tilde{J} - \tilde{A} - E_1^*)$

are both generalized adjacency matrices for the subgraph induced on the first subconstituent with respect to x.

Moreover, A^+, A^- have 0 diagonal.

Proof. Pick vertices $y, z \in X$ such that $\partial(x, y) = \partial(x, z) = 1$.

Show that A^+_{yz}, A^-_{yz} are both 0 if $\partial(y, z) = 0$ or 2.

Since $A^+_{yz} = R^{-1} E_2^* A_2 E_1^* \hat{y} - \Psi E_1^* \hat{y} = \delta^+_{11}$,

$$A^+_{yz} = \langle A^+ \hat{y}, \hat{z} \rangle = \langle \delta^+_{11}, \hat{z} \rangle = 0,$$

if $\partial(y, z) = 0$ or 2.

Since

$$A^- \hat{y} = \tilde{A} A^+ \hat{y} - \left(\frac{a_2}{c_2} - \Psi\right) E_1^* \hat{y} + \Psi(\tilde{J} - \tilde{A} - E_1^*) \hat{y} \tag{39.10}$$

$$= \tilde{A} \delta^+_{11} - \left(\frac{a_2}{c_2} - \Psi\right) \hat{y} + \Psi \delta_{12} \tag{39.11}$$

$$= \delta^-_{11}, \tag{39.12}$$

$$A^-_{yz} = \langle A^-\hat{y}, \hat{z}\rangle = \langle \delta^-_{11}, \hat{z}\rangle = 0,$$

if $\partial(y,z) = 0$ or 2.

Since $E_1^*TE_1^* = \mathrm{Span}(\tilde{J}, E_1^*, \tilde{A}, \tilde{A}^2, \ldots)$ by Lemma 33.4.

A^+, A^- are both generalized matrices for the adjacency subgraph induced on the first subconstituent concerning x. □

Similarly,

$$E_1^*TE_1^* \ni \tilde{J}, E_1^*, \tilde{A}, A^+, A^-,$$

and $\dim E_1^*TE_1^* \leq 5$.

Fact: With the above assumption,

$$E_1^*TE_1^* = \mathrm{Span}(\tilde{J}, E_1^*, \tilde{A}, A^+, A^-)$$

(may not be independent).

Lemma 39.4. *If $\partial(x,y) = 1$, then*

$$T(y)\hat{y} = T(x)\hat{y}.$$

Proof.

$$\begin{aligned}
T(x)\hat{x} &= T(x)E_1^*\hat{y} && (39.13)\\
&= M(E_0^* + E_1^*)T(x)E_1^*\hat{y} \quad \text{(as } \Gamma \text{ is thin)} && (39.14)\\
&= M\hat{x} + ME_1^*TE_1^*\hat{y} && (39.15)\\
&= M\hat{x} + M\mathrm{Span}(\tilde{J}, E_1^*, \tilde{A}, A^+, A^-)\hat{y} && (39.16)\\
&= M\hat{x} + M\mathrm{Span}(\delta_{12} + \delta_{11} + \delta_{10}, \delta_{10}, \delta_{11}, \delta_{11}^+, \delta_{11}^-) && (39.17)\\
&= M\mathrm{Span}(\delta_{01}, \delta_{10}, \delta_{11}, \delta_{11}^+, \delta_{11}^-). && (39.18)
\end{aligned}$$

But the identity of these conditions does not change if we interchange x and y.

Hence,

$$T(y)\hat{y} = T(x)\hat{y}.$$

This proves the lemma. □

Chapter 40

Structure of 1-Thin DRG

Friday, May 7, 1993

Lemma 40.1. *With the above notation, let W denota a thin irreducible T-module of endpoint 0 or 1. Pick $0 \neq v \in E_1^* V$. Then the following hold.*

(i) Eigenvalue for $\tilde{J}$ is 0 if $r(W) = 1$, and k if $r(W) = 0$.

(ii) Eigenvalue for E_1^* is 1 if $r(W) = 1$, and 1 if $r(W) = 0$.

(iii) Eigenvalue for $\tilde{A}$ is $a_0(W)$ if $r(W) = 1$, and a_1 if $r(W) = 0$.

(iv) Eigenvalue for A^+ is $a^+(W) = \frac{\gamma_1}{c_2} - 1 - \Psi$ if $r(W) = 1$, and $\frac{a_2}{c_2} - \Psi$ if $r(W) = 0$.

(v) Eigenvalue for A^- is $a^-(W) = a_0(W)\left(\frac{\gamma_1}{c_2} - 1 - 2\Psi\right) - \frac{a_2}{c_2}$ if $r(W) = 1$,

where

$$\gamma_0 = 1 + a_0(W), \text{ and } \gamma_1 = \frac{c_2 b_2 \gamma_0}{b_1 + \gamma_0(a_1 + 2 - c_2) - \gamma_0^2}$$

as in Theorem 14.2. (The eigenvalue for A^- on v will be discussed later in this lecture.)

Proof.

$(i) - (iii)$ Clear.

(iv) We have

$$A^+ = R^{-1}E_2^*A_2E_1^* - \Psi E_1^*, \tag{40.1}$$

$$A_2 = \frac{A^2 - a_1A - kI}{c_2}, \tag{40.2}$$

$$E_2^*A_2E_1^* = E_2^*\left(\frac{A^2 - a_1A - kI}{c_2}\right)E_1^* \tag{40.3}$$

$$= \frac{1}{c_2}(RF + FR - a_1R)E_1^*. \tag{40.4}$$

If $r(W) = 1$,

$$A^+v = \frac{1}{c_2}(R^{-1}RFv + R^{-1}FRv - a_1R^{-1}Rv) - \Psi v \tag{40.5}$$

$$= \frac{1}{c_2}(R^{-1}Ra_0(W)v + R^{-1}a_1(W)Rv - a_1R^{-1}Rv) - \Psi v \tag{40.6}$$

$$= \frac{1}{c_2}\left(a_0(W) + a_1(W) - a_1\right) - \Psi)\, v. \tag{40.7}$$

But,

$$a_1(W) = \gamma_1 - \gamma_0 + a_1 + 1 - c_2, \quad \gamma_0 = a_0(W) + 1$$

by Theorem 16.1.

So,

$$A^+v = \left(\frac{1}{c_2}(a_0(W) + \gamma_1 - \gamma_0 + a_1 + 1 - c_2 - a_1) - \Psi)\right) v \tag{40.8}$$

$$= \left(\frac{\gamma_1}{c_2} - 1 - \Psi\right) v. \tag{40.9}$$

If $r(W) = 0$,

$$A^+v = \frac{1}{c_2}(R^{-1}RFv + R^{-1}FRv - a_1R^{-1}Rv) - \Psi v \tag{40.10}$$

$$= \frac{1}{c_2}(R^{-1}Ra_1v + R^{-1}a_2Rv - a_1R^{-1}Rv) - \Psi v \tag{40.11}$$

$$= \left(\frac{a_2}{c_2} - \Psi\right) v. \tag{40.12}$$

(v) Immediate from (iv), and

$$A^- = \tilde{A}A^+ - \left(\frac{a_2}{c_2} - \Psi\right)E_1^* + \Psi(\tilde{J} - \tilde{A} - E_1^*).$$

HS MEMO

If $r(W) = 1$,

$$A^{-}v = \left(a_0(W) \left(\frac{\gamma_1}{c_2} - 1 - \Psi \right) - \left(\frac{c_2}{a_2} - \Psi \right) + \Psi(-a_0(W) - 1) \right) v \quad (40.13)$$

$$= \left(a_0(W) \left(\frac{\gamma_1}{c_2} - 1 - 2\Psi \right) - \frac{c_2}{a_2} \right) v. \quad (40.14)$$

If $r(W) = 0$,

$$A^{-}v = \left(a_1 \left(\frac{a_2}{c_2} - \Psi \right) - \left(\frac{a_2}{c_2} - \Psi \right) + \Psi(k - a_1 - 1) \right) v \quad (40.15)$$

$$= \left((a_1 - 1)\frac{a_2}{c_2} + (k - 2a_1)\Psi \right) v. \quad (40.16)$$

This completes the proof.

$\square$

Let W_1, W_2, W_3, W_4 denote 4 possible isomorphism classes of T-modules of endpoint 1. Then $a_0(W_1), a_0(W_2), a_0(W_3), a_0(W_4)$ are roots of a fourth degree polynomial whose coefficients are determined from intersection numbers of Γ.

So, $a_0(W_1), a_0(W_2), a_0(W_3), a_0(W_4)$ are determined by intersection numbers.

Let $\widetilde{m_i}$ denote the multiplicity of W_i $(1 \leq i \leq 4)$, which is equal to the multiplicity of $a_0(W)$ as eigenvalue 1 of $\widetilde{A}|_{(E_1^*V)_{new}}$.

Lemma 40.2. *With the above notation, we have the following.*

(i) $\tilde{m}_1$, $\tilde{m}_2$, $\tilde{m}_3$, $\tilde{m}_4$ *are determined from intersection numbers and* Ψ*.*

(ii) $\tilde{m}_i$ *is independent of vertex* x*.* $(1 \leq i \leq 4)$.

(iii) $\ell := \dim E_1^* T E_1^*$ *is independent of* x*.*

Proof.

(i) Let $e_i \in E_1^* T E_1^*$ $(1 \leq i \leq 4)$ denote the orthogonal projection on to the maximal eigenspace of $(E_1^*V)_{new}$ corresponding to λ_i. ($e = 0$ if and only if λ_i does not appear.) Set

$$e_0 = \frac{1}{k}\tilde{J}.$$

Then eigenvalues for each e_1, e_1, e_3, e_4 are as follows.

	e_0	e_1	e_2	e_3	e_4
$\tilde{J}$	k	0	0	0	0
E_1^*	1	1	1	1	1
$\tilde{A}$	a_1	$a_0(W_1)$	$a_0(W_2)$	$a_0(W_3)$	$a_0(W_4)$
A^+	$\frac{a_2}{c_2} - \Psi$	$a^+(W_1)$	$a^+(W_2)$	$a^+(W_3)$	$a^+(W_4)$
A^-	$\star$	$a^-(W_1)$	$a^-(W_2)$	$a^-(W_3)$	$a^-(W_4)$

Observe that $e_i^2 = e_i$, $\operatorname{trace} e_i = \operatorname{rank} e_i = \tilde{m}_i$ $(1 \leq i \leq 4)$, and $\operatorname{trace} e_0 = \operatorname{rank} e_0 = 1$.

By taking the trace of $\tilde{J}, E_1^*, \tilde{A}, A^+, A^-$, we have

$$
\begin{aligned}
&k = k, && (40.17)\\
&k = 1 + \tilde{m}_1 + \tilde{m}_2 + \tilde{m}_3 + \tilde{m}_4, && (40.18)\\
&0 = a_1 + a_0(W_1)\tilde{m}_1 + a_0(W_2)\tilde{m}_2 + a_0(W_3)\tilde{m}_3 + a_0(W_4)\tilde{m}_4, && (40.19)\\
&0 = \left(\frac{a_2}{c_2} - \Psi\right) + a^+(W_1)\tilde{m}_1 + a^+(W_2)\tilde{m}_2 + a^+(W_3)\tilde{m}_3 + a^+(W_4)\tilde{m}_4, && (40.20)\\
&0 = (\star) + a^-(W_1)\tilde{m}_1 + a^-(W_2)\tilde{m}_2 + a^-(W_3)\tilde{m}_3 + a^-(W_4)\tilde{m}_4. && (40.21)
\end{aligned}
$$

The coefficient matrix for $\tilde{m}_1, \tilde{m}_2, \tilde{m}_3, \tilde{m}_4$ is nonsingular (this is what you need to check and show).

HS MEMO

Complutation is not completed.

(ii) Ψ is independent of base vertex x.

(iii) We have

$$
\begin{aligned}
\dim E_1^* T E_1^* &= |\{i \mid 1 \leq i \leq 4,\ e_i \neq 0\}| + 1 && (40.22)\\
&= |\{i \mid 1 \leq i \leq 4,\ \tilde{m}_i \neq 0\}| + 1. && (40.23)
\end{aligned}
$$

This completes the proof of the lemma.

□

Let $\Gamma = (X, E)$ be thin distance regular of diameter $D \geq 5$, and Q-polynomial with respect to $E_0, E_1, \ldots, E_D$.

Fix vertices $x, y \in X$ with $\partial(x, y) = 1$,

$$E_{ij}^* \equiv E_i^*(x) E_j^*(y), \quad \delta_{ij} = E_{ij}^* \delta.$$

We saw

$$T(x)\hat{y} = T(y)\hat{x}.$$

Hence,

$$H := T(x)\hat{y} = T(y)\hat{x}$$

is a $T(x,y)$ module. $T(x,y) \subseteq \mathrm{Mat}_X(\mathbb{C})$ is generated by M, $M^*(x)$, $M^*(y)$.

Lemma 40.3. *With the above notation, we have the following.*

(i) $E^*_{i,i+1}H = \mathrm{Span}(\delta_{i,i+1}) \quad (0 \le i \le D-1)$.

(ii) $E^*_{i+1,i}H = \mathrm{Span}(\delta_{i+1,i}) \quad (0 \le i \le D-1)$.

(iii) $E^*_{i,i}H = \ell - 2 \le 3 \quad (1 \le i \le D-1)$.

Proof.

(i) $\supseteq$: We have

$$\delta_{i,i+1} = E^*_i A_{i+1}\hat{y} \in T(x)\hat{y} = H.$$

$\subseteq$: Pick $h \in E^*_{i,i+1}H$. Then $h = R^{i-1}v$, where $v = (R^{-1})^{i-1}h \in E^*_1 V$.

So, $v \in \mathrm{Span}(\delta_{12}, \delta_{11}, \delta_{10}, \delta^+_{11}, \delta^-_{11})$.

HS MEMO

$$\begin{aligned}
v &\in E^*_1 V \cap T(x)\hat{y} && (40.24)\\
&= E^*_1 T(x) E^*_1 \hat{y} && (40.25)\\
&= \mathrm{Span}(\tilde{J}, E^*_1, \tilde{A}, A^+, A^-)\hat{y} && (40.26)\\
&= \mathrm{Span}(\delta_{10} + \delta_{11} + \delta_{12}, \delta_{10}, \delta_{11}, \delta^+_{11}, \delta^-_{11}) && (40.27)\\
&= \mathrm{Span}(\delta_{10}, \delta_{11}, \delta_{12}, \delta^+_{11}, \delta^-_{11}). && (40.28)
\end{aligned}$$

Hence, there exists $\alpha \in \mathbb{C}$ such that

$$v - \alpha\delta_{12} \in \mathrm{Span}(\delta_{10}, \delta_{11}, \delta^+_{11}, \delta^-_{11}) = E^*_{11}H + E^*_{10}H.$$

So,

$$v - \alpha(\delta_{12} + \delta_{11} + \delta_{10}) \in E^*_{11}H + E^*_{10}H.$$

$$\begin{aligned}
E^*_{ii}H + E^*_{i,i-1}H &\ni R^{i-1}(v - \alpha(\delta_{12} + \delta_{11} + \delta_{10})) && (40.29)\\
&= h - \alpha'(\delta_{i,i+1} + \delta_{ii} + \delta_{i,i-1}). && (40.30)
\end{aligned}$$

Hence,

$$h - \alpha'\delta_{i,i+1} \in (E^*_{ii}H + E^*_{i,i-1}H) \cap E^*_{i,i+1}H.$$

Thus,

$$h = \alpha' \delta_{i,i+1} \in \mathrm{Span}(\delta_{i,i+1}).$$

(ii) By symmetry, we have the assertion.

(iii) $E_i^* H = E_{i,i+1}^* H + E_{i,i}^* H + E_{i,i-1}^* H$, and $\dim E_i^* H = \ell$, $\dim E_{i,i+1}^* H = 1$, and $\dim E_{i,i-1}^* H = 1$.

Hence, $\dim E_{i,i}^* H = \ell - 2$.

□

HS MEMO

Since $H = T(x)\hat{y} \subseteq T(x)E_1^*(x)V$, and

$$(R^{-1})^{i-1} : E_i^* H \to E_1^* H$$

is one-to-one and onto if $i \leq D$.

Theorem 40.1. *Let $\Gamma = (X, E)$ be thin distance regular of diameter $D \geq 5$, and Q-polynomial with respect to $E_0, E_1, \dots, E_D$.*

Pick i $(2 \leq i \leq D)$, and pick $x, y, z \in X$ such that $\partial(x, y) = 1$, $\partial(y, z) = i - 1$, $\partial(x, z) = i$.

Then,

$$z_i = |\{w \mid w \in W, \partial(x, w) = 1, \partial(y, w) = 1, \partial(z, w) = i - 1\}|$$

is independent of x, y, z.

Proof. Observe that z_i is the zx entry in

$$\Delta = E_{i-1}^*(y) A_{i-1} E_1^*(y) A E_1^*(y)$$

as

$$\Delta \hat{x} = \sum_{z \in X, \partial(x,z)=i, \partial(y,z)=i-1} z_i(x, y, z)\hat{z}.$$

Hence, $z_i(x, y, z)$ is independent of z.

So, $z_i(x, y, z)$ is determined by intersection numbers and $\Psi = \Psi(x, y)$, which is independent of x, y as well. □

Appendix A

Open Problems

Some Open Problems Concerning Distance-Regular Graphs, the Thin Condition, and the Q-Polynomial Property

Paul Terwilliger

The questions below are unsolved as of May, 1993 (to my knowledge). A complete solution (or even a significant partial solution in some cases) to any one of these problems would be publishable. I have tried to estimate the level of difficulty of each problem listed below. A $\star$ means I believe the problem is relatively easy in the sense that it can be solved using ideas from the course. There are no conceptual gaps to overcome that I am aware of (but the calculations might be quite difficult, however!). A $\star\star\star\star$ means I have no idea how to begin to attack the problem. I am only mentioning problems of this kind to give you an idea about what is known in this field.

Dist: Γ is distance-transitive.

Q: Γ is Q-polynomial with respect to the ordering $E_0, E_1, \dots, E_D$ of the primitive idempotents.

Bip: Γ is bipartite.

Th: Γ is thin (over the field of complex numbers).

Few1: The subgraph induced on the first subconstituent of Γ with respect to x

has at most 5 distince eigenvalues.

Few2: The subgraph induced on the second subconstituent of Γ with respect to x has at most 16 distinct eigenvalues.

Z: For all integers i $(2 \leq i \leq D)$, and all triples u, v, w $(u, v, w \in X)$ such that $\partial(u, v) = 1$, $\partial(v, w) = i - 1$, and $\partial(v, w) = i$, the number

$$z_i := |\{y \mid y \in X, \partial(y, u) = \partial(y, v) = 1, \partial(y, w) = i - 1\}|$$

is a constant that does not depend on u, v, w.

The following implications are known:

$$Q + Bip \rightarrow TH, \qquad Q + TH \rightarrow Few1, Few2, Z.$$

(1) ⋆⋆⋆⋆ Classify all the distance-regular graphs (with sufficiently large diameter). If necessary, assume some combination of the above properties. (My personal goal is to classify all the graphs Γ satisfying *Q*, *TH*. I expect this will take a number of years.)

(2) ⋆⋆ Assume *Q*, *Bip*, and classify Γ.

(3) ⋆ Find generalization to the theorems of the course for non-regular, bipartite distance-regular graphs.

(4) ⋆ Assume, *Q*, and let W denote an irreducible T-module with endpoint 1 that is not thin. Find a nice basis for W and find the matrices representing the adjacency matrix A and the dual adjacency matrix A^* with respect to this basis. Perhaps assume classical parameters. Theorem 30.1, and Lemma 31.1 should be useful.

(5) ⋆ Is it true that Γ is thin over the field of complex numbers if and only if Γ is thin over the field of real numbers? What does it mean for Γ to be thin over the field of rational numbers? The examples suggest that if Γ is thin over the complex numbers then it is already thin over the rational numbers. If this is true, it would be nice to have a proof. For the moment, suppose it is not true. Assume Γ is thin over the field of complex numbers, and define the *splitting field* of Γ to be the minimal extension of the rational field over which Γ is thin. Then the elements of the Galois group of the splitting field act on the standard module, and permute the isomorphism classes of irreducible T-modules. How are the isomorphism classes of T-modules involved related? Can the permutations be nontrivial?

(6) $\star\star$ Assume Q, and assume there is a second Q-polynomial ordering of the primitive idempotent. Prove *TH*. I believe in this case the first subconstituent has at most 4 distinct eigenvalues, and the constant Ψ from class if determined by the intersection numbers. It may be possible to classify all such Γ.

(7) $\star\star$ Assume Q, and assume there is a second P-polynomial ordering of the distance matrices. I believe the same thing happens as in (6) above.

(8) $\star\star$ A path $y = y_0, y_1, \ldots, y_t = z$ in Γ is said to be *geodetic* whenever $\partial(y, z) = t$. Let us say a subset Δ of X is *geodetically closed* whenever all vertices on all geodetic paths with endpoints in Δ are also in Δ. For any vertices $y, z \in X$, observe there exists a unique minimal geodetically closed subset containing y, z, denoted $[yz]$.

If the diameter of $[yz]$ equals $\partial(y, z)$, we say $[yz]$ is a subspace. Furthermore, show the subgraph induced on $[yz]$ is distance-regular, and satisfies Q, *TH*. If this proves not to be the case, find a simple additional assumption on Γ under which it is true. (It seems to hold for the known examples). I believe these subspaces are the key to an eventual classification of the graphs satisfying Q, *TH* (and possibly all distance-regular graphs with sufficiently large diameter). In the examples, the partially ordered set of all subspaces, ordered by reverse inclusion, is some classical geometry. There are many classification theorems in the area of finite projective geometry. My hope is that given any Γ, the partially ordered set of all subspaces is some highly regular geometry that can be classified using one of these theorems, leading us to a classification of the original Γ. (By the way, I intend to explore this area in the course I am teaching next fall on partiallly ordered sets).

(9) $\star\star$ Assume Q, *TH*. Find a nice basis for $E_2^*TE_2^*$ in a way that generalized what we did in class for $E_1^*TE_1^*$.

(10) $\star$ Assume B, *TH*, and that the dimension of $E_2^*TE_2^*$ is at most 4. Show that Q holds. Find a nice basis for $E_2^*TE_2^*$.

(11) It is not hard to show that in general

$$c_i \geq c_{i-1} \quad (1 \leq i \leq D), \tag{A.1}$$
$$b_i \leq b_{i-1} \quad (0 \leq i \leq D-1). \tag{A.2}$$

It is known that if Γ has at least one cyle $y1, y2, y3, y4, y1$ such that $\partial(y1, y3) = \partial(y2, y4) = 2$ then

$$c_i - c_{i-1} + b_{i-1} - b_i \geq a_1 + 2 \quad (1 \leq i \leq D).$$

This bound has proved to be quite fndamental. For example, the graphs Γ where equality holds for all i all satisfy Q, and in fact they are precisely the graphs of type IIA or IIC (refereng to p.10, 11 in the thick paper I handed out in class). These graphs have all been classified. I have some papers describing some more general bounds of the above sort, but they are unsatisfactory in the sense that the class of graphs for which equality is attained is not interesting, and may even be empty. Hence one problem ($\star\star$) is to find a bound that controls the growth of the c_i's and the decrease of the b_i's, where equality is attained for some nice, large class of graphs. Ideally, this class would contain all the known examples of Γ with sufficiently large diameter, or perhaps all the graphs Γ satisfying $Q + TH$. Specific proble ($\star$): Assume Z and redo the arguments in the above-mentioned papers. Dramatic improvements in the bounds obtained are expected (I did not realise the significance of Z and redo the arguments in the above-mentioned papers). Since $Q + TH \to Z$, the new bounds are expected to give important feasibility conditions on the intersection numbers of any Γ satisfying Q and TH.

(12) $\star$ Explore the class of graphs that are Q-polynomial with respect to each vertex. but not assumed to be distance-regular. Are these graphs in fact distance-regular or bi-distance-regular? (This result would be very esthetically pleasing to me, since as we have seen, the sibling property of being thin does not imply distance-regularity or bi-distance-regularity). If the answer to the above question is "no", just what sort of regularity do these graphs have? For a graph that is Q-polynomial with respect to each vertex, how must the orderings of the primitive idempotents associated with adjacent vertices be related? Is it possible for a distance-regular graph to be Q-polynomial with respect to each vertex, but still not be Q-polynomial? (This is a completely new area. Up until now, the Q-polynomial property was only defined for distance-regular graphs.)

(13) $\star\star$ To what extent do the polynomial relations on R, L, F given in Theorem 30.1 actually characterize the Q-polynomial property? For example, suppose

(*i*) $L^2FE_i^*, LFLE_i^*, FL^2E_i^*, L^2E_i^*$ are linearly dependent for all i $(2 \leq i \leq D)$.

(*ii*) $FLRE_i^*, FRLE_i^*$ are linearly dependent for all i $(0 \leq i \leq D)$, and

(*iii*) $RL^2E_i^*, LRLE_i^*, L^2RE_i^*, LF^2E_i^*, FLFE_i^*, LFE_i^*, F^2LE_i^*, FLE_i^*, LE_i^*$ are linearly dependent, for all i $(1 \leq i \leq D)$.

Then does Q hold? what if we assume TH? If not, what other graphs can one get? are they "almost" Q-polynomial in some sense (pserhaps many Krein parameters vanish, but not quite enough to imply Q). What is the essential assumption about the coefficients in the above dependencies that is needed to insure Q.

(14) $\star\star\star$ Assume Q and TH. Find the abstract structure of the Norton algebra N. My intuition says that this structure can be computed in terms of the

intersection numbers and a small list of additional parameters such as ψ. The examples suggest that N is "almost associative" in some sense. Specific problem ($\star$) Find the precise structure of the Norton algebra for the examples $J(d,n)$, $J_q(d,n)$, ..., and find some pattern. The dual of Theorem 30.1 is relevent to this problem. My intuition says that the idempotents of N should correspond to the subspaces of Γ referred to in problem 8, and that somehow the multiplication operation in N should be related to the meet and join operations in the geometry of subspaces referred to in that problem.

(15) $\star\star$ Assume Q and *TH*, and pick $y \in X$. Show

$$T(x)\hat{y} = T(y)\hat{x}.$$

(I can show this for $\partial(x,y) = 1$.) If the above line holds, then apparently $H := T(x)\hat{y} = T(y)\hat{x}$ is a module for the algebra $T(x,y)$ generated by the Bose-Mesner algebra M, the dual Bose-Mesner algebra $M^*(x)$, and $M^*(y)$. Observe the elements of $M^*(x)$, $M^*(y)$ mutually commute, and in fact that the maximal common engenspaces of $M^*(x)$, $M^*(y)$ are the $E^*_{ij}V$ $(0 \le i,j \le D)$, where $E^*_{ij} = E^*_i(x)E^*_j(y)$. Find a nice orthogonal basis for each $E^*_{ij}H$. Observe the union B of these bases is a basis for H. Find the matrices representing A, $A^*(x)$, $A^*(y)$ with respect to B. Choose B so that the entries in these matrices are nice, factorable expressions in the intersection numbers and whatever other parameters are needed. In the case $\partial(x,y) = 1$, these entries can be deteermined from the intersection numbers and the parameter ψ. If $\partial(x,y) \ge 2$, presumably there are some more free parameters analoguous to ψ that play a role. My intuition says that as a $T(x,y)$-module, H is determined from the intersection numbers of Γ and t free parameters, where $t = \partial(x,y)$.

(16) $\star\star$ Does *TH* and *Few1* imply Z? If not, what extra assumption is needes?

(17) $\star\star$ Does *TH*, *Few1*, *Few2*, imply Q? If not, what extra assumption is needed?

(18) $\star\star$ Let Γ be an arbitrary grarph, not assumed to be distance-regular. Conjecture: Γ is thin if and only if for all integers i,j,k, and all vertices $x,y,z \in X$ such that $\partial(x,y) = \partial(x,z) = i$, the number of vertices $w \in X$ with $\partial(w,x) = j$, $\partial(w,y) = 1$, $\partial(w,z) = k$ equals the number of vertices $w' \in X$ with $\partial(w',x) = j$, $\partial(w',z) = 1$, $\partial(w',y) = k$. If Γ assumed to be distance-regular, then the conjecrure is true and there is a long proof in the thick paper I handed out in class (Theorem 5.1 (iii)) . A short, slick proof (assuming distance-regularity or not) is very much needed. If the conjecture turns out not to be true in the bi-distance-regular case, find some similar combinatorial characterization of the thin property.

There are a number of additional problems in section 7 of the thick paper I handed out in class. Essentially all the known examples of thin, Q-polynomial distance-regular graphs are listed in section 6 of that paper.

For each of the above problems, I have a good deal of background information to communicate, but unfortunately in most cases it is not in published form! If you tell me what problem you want to focus on, I can tailor a series of lectures this summer towards communicating what I know on the subject. But one key point: Often "I don't know what I know". If you are constantly asking probing questions of me it makes my job a lot easier: it often reminds me of information that is relevant that I had forgotten, or that I had forgotten was relevant.

Appendix B

Comparison Table

We list Definitions, Theorems, Lemmas, etc. with the numbers in the original handwritten note.

Chapter	New Numbering	Old Numbering
1	Example 1.1	Example
	Example 1.2	Example
	Definition 1.1	Definition
	Lemma 1.1	Lemma 1
	Definition 1.2	Definition
	Definition 1.3	Definition
	Definition 1.4	Definition
	Definition 1.5	Definition
	Definition 1.6	Definition
	Definition 1.7	Definition
	Lemma 1.2	Lemma 2
2	Definition 2.1	Definition
	Definition 2.2	Definition
	Theorem 2.1	Theorem 3
	Lemma 2.1	Lemma 4
	Definition 2.3	Definition
	Corollary 2.1	Corollary 5
3	Definition 3.1	Definition
	Definition 3.2	Definition
	Definition 3.3	Definition
	Definition 3.4	Definition
	Example 3.1	Example
	Example 3.2	Example
	Example 3.3	Example

Chapter	New Numbering	Old Numbering
	Theorem 3.1	Theorem 6
	Definition 3.5	Definition
	Example 3.4	Example
	Lemma 3.1	Lemma 7
4	Theorem 4.1	Theorem 8
	Example 4.1	Example
	Example 4.2	Example
	Definition 4.1	Definition
	Lemma 4.1	Lemma 9
5	Definition 5.1	Definition
	Theorem 5.1	Theorem 10
6	Theorem 6.1	Theorem 11
	Definition 6.1	Definition
	Definition 6.2	Definition
7	Definition 7.1	Definition
	Example 7.1	Example
	Lemma 7.1	Lemma 12
	Theorem 7.1	Theorem 13
8	Lemma 8.1	Lemma 14
9	Lemma 9.1	Lemma 15
	Corollary 9.1	Corollary 16
	Lemma 9.2	Lemma 17
	Definiton 9.2	Definition
10	Lemma 10.1	Lemma 18
	Lemma 10.2	Lemma 19
	Corollary 10.1	Corollary 20
11	Lemma 11.1	Lemma 21
	Lemma 11.2	Lemma 22
12	Lemma 12.1	Lemma 23
	Theorem 12.1	Theorem 24
13	Lemma 13.1	Lemma 25
	Theorem 13.1	Theorem 26
	Proposition 13.1	Proposition 27
14	Lemma 14.1	Lemma 28
	Lemma 14.2	Lemma 29
15	Definition 15.1	Definition
	Lemma 15.1	Lemma 30
16	Definition 16.1	Definition
	Lemma 16.1	Lemma 31
	Theorem 16.1	Theorem 32
	Lemma 16.2	Lemma 33*
17	Definition 17.1	Definition
	Definition 17.2	Definition

Chapter	New Numbering	Old Numbering
	Example 17.1	Example 1
	Example 17.2	Example 2
	Exercise 17.1	Exercise
	Example 17.3	Example 3
18	Lemma 18.1	Lemma 33
19	Lemma 19.1	Lemma 34
	Definition 19.1	Definition:
20	Lemma 20.1	Lemma 34-a
	Lemma 20.2	Lemma 34-b
	Lemma 20.3	Lemma 35
	Corollary 20.1	Corollary 36
	Lemma 20.4	Lemma 37
21	Lemma 21.1	Lemma 38
	Lemma 21.2	Lemma 39
22	Lemma 22.1	Lemma 40
	Definition 22.1	Definition
	Lemma 22.2	Lemma 41
23	Theorem 23.1	Theorem 42
	Definition 23.1	Definition
	Example 23.1	Example
24	Definition 23.2	Definition
	Lemma 23.1	Lemma 43
	Definition 24.1	Definition
	Theorem 24.1	Theorem 44
26	Corollary 26.1	Corollary 45
	Lemma 26.1	Lemma 46
27	Theorem 27.1	Theorem 47
	Definition 27.1	Definition
	Definition 27.2	Definition
	Lemma 27.1	Lemma 48
	Example 27.1	Example
28	Lemma 28.1	Lemma 49
	Conjecture 28.1	Conjecture
29	Theorem 29.1	Theorem 50
30	Theorem 30.1	Theorem 51
	Lemma 30.1	Lemma 52
	Corollary 30.1	Corollary 53
31	Lemma 31.1	Lemma 54
32	Lemma 32.1	Lemma 55
	Lemma 32.2	Lemma 56
	Lemma 32.3	Lemma 57
33	Lemma 33.1	Lemma 58
	Lemma 33.2	Lemma 59

Chapter	New Numbering	Old Numbering
	Lemma 33.3	Lemma 60
	Lemma 33.4	Lemma 61
34	Lemma 34.1	Lemma 62
	Lemma 34.2	Lemma 63
	Lemma 34.3	Lemma 64
35	Theorem 35.1	Theorem 65
36	Conjecture 36.1	Conjecture
	Conjecture 36.2	Conjecture
	Conjecture 36.3	Conjecture
37	Lemma 37.1	Lemma 66
	Definition 37.1	Definition
	Example 37.1	Example
	Lemma 37.2	Lemma 67
38	Lemma 38.1	Lemma 68
	Definition 38.1	Definition
	Lemma 38.2	Lemma 69
39	Lemma 39.1	Lemma 70
	Lemma 39.2	Lemma 71
	Lemma 39.3	Lemma 72
	Lemma 39.4	Lemma 73
40	Lemma 40.1	Lemma 74
	Lemma 40.2	Lemma 75
	Lemma 40.3	Lemma 76
	Theorem 40.1	Theorem 77

Appendix C

Technical Memo

This note is created by `bookdown` package on RStudio.

For `bookdown` See (Xie, 2015), (Xie, 2017), (Yihui Xie, 2018).

The following is a memo.

A. Install R and R Studio with necessary packages if needed

B. Create and setup ssh key by `ssh-keygen`

C. Setup Git-GitHub connection

1. Create a GitHub account if needed

2. Set ssh key by copying the value of the public SSH key to the clipboard using `pbcopy` and paste it into SSH Keys in the GitHub account

D. Remote Repository

1. Log-in to the GitHub account
2. Go to RStudio/bookdown-demo repository: https://github.com/rstudio/bookdown-demo
3. Use This Template
4. Input Repository Name
5. Select Public - default
6. Create a repository from the template
7. Set Pages: Branch main, docs

E. Local Repository

1. Copy: Code > Clone > SSH from the GitHub repository
2. Create a new project by Version Control Git
3. Change directory name `_book` to `docs`
4. Edit YAMLs

All source files are in the GitHub Repository.

C.1 To Do List

- Environment `align` in ePub_book.
  - It may be better to give up ePub book mode.
- https://github.com/rstudio/bookdown/issues/530
- See also bookdown ePub version page 33. I could not retrieve the same. (See page 32 as well.)
- Environment of align
  1. `align`

  $$\begin{aligned} A &= B && \text{(C.1)} \\ &= C && \text{(C.2)} \end{aligned}$$

  2. `eqnarray*`

  $$\begin{array}{rcl} A & = & B \\ & = & C \end{array}$$

  3. `array` in equation with minus spacing

  $$\begin{array}{rcl} A & = & B \\ & = & C \end{array}$$

  4. `split` in equation

  $$\begin{split} A &= B \\ &= C \end{split} \tag{C.3}$$

- Shaded Box using `frame` with environment `hs` in PDF
- Controlling top icons
- My template of `bookdown`

Minor

- Difference in numbering; HTML and PDF
- `bs4_book` format
- `bookdown` template and `doc` directory
- Style of citation in PDF

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