Quiz 1

ID#: **Name:**

Let p, q and r be statements, and x, y and z compound statements of p, q and r. Let

$$s \equiv \neg(p \Rightarrow (q \vee r)), \quad t \equiv (p \Rightarrow q) \vee r.$$

1. Complete the truth table of s and t. Clarify the columns of the truth values of the statements.

p	q	r	$\neg \quad (p \quad \Rightarrow \quad (q \quad \vee \quad r))$	$(p \quad \Rightarrow \quad q) \quad \vee \quad r$	x	y	z
T	T	T			T	F	F
T	T	F			T	F	F
T	F	T			T	F	T
T	F	F			T	F	F
F	T	T			T	F	F
F	T	F			F	T	T
F	F	T			T	F	F
F	F	F			T	F	F

2. Choose the correct one.

   (a) $s \equiv t$ (b) $s \equiv \neg t$ (c) Neither (a) nor (b).

3. Express $s \equiv \neg(p \Rightarrow (q \vee r))$ using $\neg$, $\wedge$ and parentheses only. Show work or give a brief explanation.

4. Fill each underlined blank with $\neg$, $\wedge$ or $\vee$ to express x, y and z in the truth table above. There may be voids.

$$x \equiv ((\_\_\ p) \Rightarrow (\_\_\ q)) \vee (\_\_\ r),$$ *Hint: Compare with t in 1.*

$$y \equiv ((\_\_\ p)\ \_\_\ (\_\_\ q))\ \_\_\ (\_\_\ r),$$

$$z \equiv y\ \_\_(((\_\_\ p) \wedge (\_\_\ q))\ \_\_\ (\_\_\ r)).$$

Message: What is your dream? Describe your vision of yourself and the world 25 years from now. 将来の夢、25 年後の自分について、世界について。(If you don't want your message to be posted, write "Do Not Post."「HP 掲載不可」は明記の事。)

Solutions to Quiz 1

Let p, q and r be statements, and x, y and z compound statements of p, q and r. Let

$$s \equiv \neg(p \Rightarrow (q \vee r)), \quad t \equiv (p \Rightarrow q) \vee r.$$

1. Complete the truth table of s and t. Clarify the columns of the truth values of the statements.

p	q	r	$\neg$	$(p$	$\Rightarrow$	$(q$	$\vee$	$r))$	$(p$	$\Rightarrow$	$q)$	$\vee$	r	x	y	z
T	T	T	**F**	T	T	T	T	T	T	T	T	**T**	T	T	F	F
T	T	F	**F**	T	T	T	T	F	T	T	T	**T**	F	T	F	F
T	F	T	**F**	T	T	F	T	T	T	F	F	**T**	T	T	F	T
T	F	F	**T**	T	F	F	F	F	T	F	F	**F**	F	T	F	F
F	T	T	**F**	F	T	T	T	T	F	T	T	**T**	T	T	F	F
F	T	F	**F**	F	T	T	T	F	F	T	T	**T**	F	F	T	T
F	F	T	**F**	F	T	F	T	T	F	T	F	**T**	T	T	F	F
F	F	F	**F**	F	T	F	F	F	F	T	F	**T**	F	T	F	F

2. Choose the correct one.

   (a) $s \equiv t$ $\boxed{\text{(b) } s \equiv \neg t}$ (c) Neither (a) nor (b).

3. Express $s \equiv \neg(p \Rightarrow (q \vee r))$ using $\neg$, $\wedge$ and parentheses only. Show work or give a brief explanation.

   Solution.

$$\begin{aligned} s &\equiv \neg(p \Rightarrow (q \vee r)) \\ &\equiv \neg(\neg p \vee (q \vee r)) \qquad \text{Problem 1 (f) in Handout} \\ &\equiv \neg(\neg p) \wedge \neg(q \vee r) \qquad \text{Problem 1 (e)} \\ &\equiv p \wedge (\neg q \wedge \neg r) \qquad \text{Problem 1 (a), (e)} \\ &\equiv p \wedge \neg q \wedge \neg r \qquad \text{Problem 1 (c).} \end{aligned}$$

   Solution 2. Since the truth value of s is T only when the truth values of p, q, r are T, F, F, i.e., the truth values of $p, \neg q, \neg r$ are T, T, T, we have

$$s \equiv p \wedge \neg q \wedge \neg r.$$

4. Fill each underlined blank with $\neg$, $\wedge$ or $\vee$ to express x, y and z in the truth table above. There may be voids.

$$\begin{aligned} x &\equiv ((\underline{\neg}\ p) \Rightarrow (\underline{\neg}\ q)) \vee (\underline{\quad} r), \qquad \textit{Hint: Compare with } t \textit{ in 1.} \\ y &\equiv ((\underline{\neg}\ p)\ \underline{\wedge}\ (\underline{\quad} q))\underline{\wedge}\ (\underline{\neg}\ r), \\ z &\equiv y \underline{\vee}\ (((\underline{\quad} p) \wedge (\underline{\neg}\ q))\underline{\wedge}\ (\underline{\quad} r)). \end{aligned}$$

Quiz 2

ID#: **Name:**

Let $[i,j;c], [i,j], [i;c]$ be the following elementary row operations. (1) $[i,j;c]$: Replace row i by the sum of row i and c times row j. (2) $[i,j]$: Interchange row i and row j. (3) $[i;c]$: Multiply all entries in row i by a nonzero constant c.

We applied elementary row operations to the augmented matrix A of a system of linear equations.

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 & -3 & 0 & 2 \\ 5 & -10 & 3 & 0 & 3 & -6 & -20 \\ 0 & 0 & 2 & 0 & 12 & -3 & -15 \\ -1 & 2 & 0 & 1 & -2 & 0 & -1 \end{bmatrix} \xrightarrow{(a)} \begin{bmatrix} 1 & -2 & 0 & 0 & -3 & 0 & 2 \\ 0 & 0 & 3 & 0 & 18 & -6 & -30 \\ 0 & 0 & 2 & 0 & 12 & -3 & -15 \\ -1 & 2 & 0 & 1 & -2 & 0 & -1 \end{bmatrix} \xrightarrow{(b)}$$

$$\begin{bmatrix} 1 & -2 & 0 & 0 & -3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 6 & -2 & -10 \\ 0 & 0 & 2 & 0 & 12 & -3 & -15 \\ -1 & 2 & 0 & 1 & -2 & 0 & -1 \end{bmatrix} \xrightarrow{(c)} \begin{bmatrix} 1 & -2 & 0 & 0 & -3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 6 & -2 & -10 \\ 0 & 0 & 2 & 0 & 12 & -3 & -15 \\ 0 & 0 & 0 & 1 & -5 & 0 & 1 \end{bmatrix}$$

1. Write elementary row operations at (a), (b), (c) in the form $[i,j;c], [i,j]$, or $[i;c]$. (For example, (a) $[1,2;3]$, (b) $[1,2]$, (c) $[1;-2]$.)

   (a) ☐ (b) ☐ (c) ☐

2. Find the reduced echelon form of A by applying elementary row operations to the fourth matrix above. <u>Show work!</u>

3. Find the rank of the augmented matrix A of the system of linear equations, and the rank of the coefficient matrix C.

   (a) rank $A =$ (b) rank $C =$

4. Find the solutions assuming that $x_1, x_2, x_3, x_4, x_5, x_6$ are the unknowns.

Message: What is most precious to you? あなたにとって一番たいせつな（または、たいせつにしたい）もの、ことはなんですか。(If you don't want your message to be posted, write "Do Not Post."「HP 掲載不可」は明記の事。)

Solutions to Quiz 2

Let $[i,j;c]$, $[i,j]$, $[i;c]$ be the following elementary row operations. (1) $[i,j;c]$: Replace row i by the sum of row i and c times row j. (2) $[i,j]$: Interchange row i and row j. (3) $[i;c]$: Multiply all entries in row i by a nonzero constant c.

We applied elementary row operations to the augmented matrix A of a system of linear equations.

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 & -3 & 0 & 2 \\ 5 & -10 & 3 & 0 & 3 & -6 & -20 \\ 0 & 0 & 2 & 0 & 12 & -3 & -15 \\ -1 & 2 & 0 & 1 & -2 & 0 & -1 \end{bmatrix} \xrightarrow{(a)} \begin{bmatrix} 1 & -2 & 0 & 0 & -3 & 0 & 2 \\ 0 & 0 & 3 & 0 & 18 & -6 & -30 \\ 0 & 0 & 2 & 0 & 12 & -3 & -15 \\ -1 & 2 & 0 & 1 & -2 & 0 & -1 \end{bmatrix} \xrightarrow{(b)}$$

$$\begin{bmatrix} 1 & -2 & 0 & 0 & -3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 6 & -2 & -10 \\ 0 & 0 & 2 & 0 & 12 & -3 & -15 \\ -1 & 2 & 0 & 1 & -2 & 0 & -1 \end{bmatrix} \xrightarrow{(c)} \begin{bmatrix} 1 & -2 & 0 & 0 & -3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 6 & -2 & -10 \\ 0 & 0 & 2 & 0 & 12 & -3 & -15 \\ 0 & 0 & 0 & 1 & -5 & 0 & 1 \end{bmatrix}$$

1. Write elementary row operations at (a), (b), (c) in the form $[i,j;c]$, $[i,j]$, or $[i;c]$. (For example, (a) $[1,2;3]$, (b) $[1,2]$, (c) $[1;-2]$.)

   (a) [2,1;-5]　　(b) [2;1/3]　　(c) [4,1;1]

2. Find the reduced echelon form of A by applying elementary row operations to the fourth matrix above. Show work!

$$\begin{bmatrix} 1 & -2 & 0 & 0 & -3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 6 & -2 & -10 \\ 0 & 0 & 2 & 0 & 12 & -3 & -15 \\ 0 & 0 & 0 & 1 & -5 & 0 & 1 \end{bmatrix} \xrightarrow{[3,2;-2]} \begin{bmatrix} 1 & -2 & 0 & 0 & -3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 6 & -2 & -10 \\ 0 & 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 & -5 & 0 & 1 \end{bmatrix} \xrightarrow{[3,4]}$$

$$\begin{bmatrix} 1 & -2 & 0 & 0 & -3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 6 & -2 & -10 \\ 0 & 0 & 0 & 1 & -5 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 5 \end{bmatrix} \xrightarrow{[2,4;2]} \begin{bmatrix} 1 & -2 & 0 & 0 & -3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 1 & -5 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 5 \end{bmatrix}$$

   The thirs matrix is not in reduced row echelon form as it does not satisfy condition 4.

3. Find the rank of the augmented matrix A of the system of linear equations, and the rank of the coefficient matrix C.

   Solution. The rank of a matrix is the number of nonzero rows in the reduced row echelon form of the matrix. The coefficient matrix is the matrix obtained from the augmented matrix by eliminating the last column.

   (a) rank $A = 4$　　(b) rank $C = 4$

4. Find the solutions assuming that $x_1, x_2, x_3, x_4, x_5, x_6$ are the unknowns.

   Solution. Since rank $A =$ rank $C = 4$, this system is consistent. In this case there are 6 unknowns, we need $2 = 6 - 4 = 6 - \text{rank } A$ free parameters. Since the second and the fifth column in the coefficient matrix do not have leading one's, we set $x_2 = t, x_5 = u$ to be free parameters.

$$\left\{ \begin{array}{rcl} x_1 - 2x_2 - 3x_5 & = & 2 \\ x_3 + 6x_5 & = & 0 \\ x_4 - 5x_5 & = & 1 \\ x_6 & = & 5 \end{array} \right. \quad \text{より} \quad \left\{ \begin{array}{rcl} x_1 & = & 2t + 3u + 2 \\ x_2 & = & t \\ x_3 & = & -6u \\ x_4 & = & 5u + 1 \\ x_5 & = & u \\ x_6 & = & 5. \end{array} \right.$$

   If the matrix is in reduced echelon form and parameters are taken properly, $x_1, x_2, \ldots, x_6$ can be expressed by the sum of a constant and scalar multiple of parameters.

Quiz 3

ID#: **Name:**

$$A = \begin{bmatrix} 2 & 1 & -5 \\ 1 & 0 & -3 \\ 4 & 2 & -11 \end{bmatrix}, \ \boldsymbol{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \ \boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ C = \begin{bmatrix} 2 & 1 & -5 & 1 & 0 & 0 \\ 1 & 0 & -3 & 0 & 1 & 0 \\ 4 & 2 & -11 & 0 & 0 & 1 \end{bmatrix}$$

Let A, $\boldsymbol{x}$, $\boldsymbol{b}$ and C be as above, where $C = [A, I]$. We will find the inverse of A.

$$C \to C_1 = \begin{bmatrix} 0 & 1 & 1 & 1 & -2 & 0 \\ 1 & 0 & -3 & 0 & 1 & 0 \\ 4 & 2 & -11 & 0 & 0 & 1 \end{bmatrix} \to C_2 \to C_3 = \begin{bmatrix} 1 & 0 & -3 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & -2 & 0 \\ 0 & 2 & 1 & 0 & -4 & 1 \end{bmatrix} \to \cdots$$

The matrices C_1, C_2 and C_3 are obtained from C, C_1 and C_2 respectively by applying an elementary row operation once. Let S, T and U be 3×3 elementary matrices satisfying $SC = C_1$, $TC_1 = C_2$ and $UC_2 = C_3$.

1. Find the inverse of S.

2. Find the product UT of matrices U and T.

3. Find the inverse A^{-1} of A.

4. Suppose $A\boldsymbol{x} = \boldsymbol{b}$. Find x, y, z using the inverse of A.

Messages: Anything that made you rejoice, sad or angry, or you are thankful of recently? 最近とても嬉しかった（感謝している）こと、悲しかったこと、怒っていること。(If you don't want your message to be posted, write "Do Not Post."「ホームページ掲載不可」は明記のこと。)

Solutions to Quiz 3

$$A = \begin{bmatrix} 2 & 1 & -5 \\ 1 & 0 & -3 \\ 4 & 2 & -11 \end{bmatrix},\ \boldsymbol{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix},\ \boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ C = \begin{bmatrix} 2 & 1 & -5 & 1 & 0 & 0 \\ 1 & 0 & -3 & 0 & 1 & 0 \\ 4 & 2 & -11 & 0 & 0 & 1 \end{bmatrix}$$

Let A, $\boldsymbol{x}$, $\boldsymbol{b}$ and C be as above, where $C = [A, I]$. We will find the inverse of A.

$$C \to C_1 = \begin{bmatrix} 0 & 1 & 1 & 1 & -2 & 0 \\ 1 & 0 & -3 & 0 & 1 & 0 \\ 4 & 2 & -11 & 0 & 0 & 1 \end{bmatrix} \to C_2 \to C_3 = \begin{bmatrix} 1 & 0 & -3 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & -2 & 0 \\ 0 & 2 & 1 & 0 & -4 & 1 \end{bmatrix} \to \cdots$$

The matrices C_1, C_2 and C_3 are obtained from C, C_1 and C_2 respectively by applying an elementary row operation once. Let S, T and U be 3×3 elementary matrices satisfying $SC = C_1$, $TC_1 = C_2$ and $UC_2 = C_3$.

Solution. The elementary row operations applied are $[1, 2; -2]$, $[1, 2]$ and $[3, 1; -4]$. The second and the third can be $[3, 2; -4]$ and $[1, 2]$. Therefore, $S = E(1, 2; -2)$ and $T = E(1, 2)$ and $U = E(3, 1; -4)$. (Or, $T = E(3, 2; -4)$ and $U = E(1, 2)$.)

1. Find the inverse of S.

   Solution.

$$S^{-1} = E(1, 2; -2)^{-1} = E(1, 2; 2) = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \text{or}$$

$$[S, I] = [E(1, 2; -2), I] = \begin{bmatrix} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} = [I, E(1, 2; 2)].$$

2. Find the product UT of matrices U and T.

   Solution.

$$UT = E(3, 1; -4)E(1, 2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -4 & 1 \end{bmatrix} = E(1, 2)E(3, 2; -4).$$

   Note that the UT is the same for both cases.

3. Find the inverse A^{-1} of A.

   Solution. $C \to C_3 \to$

$$\begin{bmatrix} 1 & 0 & -3 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & -2 & 0 \\ 0 & 0 & -1 & -2 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 2 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 6 & 1 & -3 \\ 0 & 1 & 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 2 & 0 & -1 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & 0 & 6 & 1 & -3 \\ 0 & 1 & 0 & -1 & -2 & 1 \\ 0 & 0 & 1 & 2 & 0 & -1 \end{bmatrix} = [I, A^{-1}]. \quad \text{Hence, } A^{-1} = \begin{bmatrix} 6 & 1 & -3 \\ -1 & -2 & 1 \\ 2 & 0 & -1 \end{bmatrix}.$$

   Corresponding elementary row operations applied to C_3 are $[3, 2; -2]$, $[3; -1]$, $[1, 3; 3]$, $[2, 3; -1]$ in this order.

4. Suppose $A\boldsymbol{x} = \boldsymbol{b}$. Find x, y, z using the inverse of A.

   Solution. Since $\boldsymbol{x} = I\boldsymbol{x} = A^{-1}A\boldsymbol{x} = A^{-1}\boldsymbol{b}$, by multiplying A^{-1} to $\boldsymbol{b}$, we have the following.

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}\boldsymbol{b} = \begin{bmatrix} 6 & 1 & -3 \\ -1 & -2 & 1 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix},\ \begin{cases} x &= 4, \\ y &= -2, \\ z &= 1. \end{cases}$$

Quiz 4

ID#: **Name:**

Let $f(x) = 2x^3 - 3x^2 - 20x + 38$.

1. Find a polynomial $q(x)$ and a number r satisfying $f(x) = q(x)(x-2) + r$. Show work.

2. Find a, b, c, d satisfying $f(x) = a + b(x-2) + c(x-2)^2 + d(x-2)^3$. Show work.

3. Find a polynomial $Q(x)$ of degree 3 such that $Q(2) = 0$, $Q(3) = 1$, $Q(4) = 0$, $Q(5) = 0$.

4. Let $h(x) = a(x-3)(x-4)(x-5) + b(x-2)(x-4)(x-5) + c(x-2)(x-3)(x-4) + d(x-2)(x-3)(x-5)$. Find a, b, c, d when $h(2) = 12$, $h(3) = 1$, $h(4) = -2$, $h(5) = 6$. Show work.

5. Find a polynomial of degree 7 satisfying $g(2) = 12$, $g(3) = 1$, $g(4) = -2$, $g(5) = 6$.

Messages: What kind of adult is admirable? What is admirable about children? どんなおとなが魅力的ですか。こどもの魅力は何でしょう。(If you don't want your message to be posted, write "Do Not Post."「HP 掲載不可」は明記の事。)

Solutions to Quiz 4

Let $f(x) = 2x^3 - 3x^2 - 20x + 38$.

1. Find a polynomial $q(x)$ and a number r satisfying $f(x) = q(x)(x-2) + r$. Show work.

   Solution. Applying synthetic division

$$\begin{array}{c|cccc} 2 & 2 & -3 & -20 & 38 \\ & & 4 & 2 & -36 \\ \hline & 2\ (c_2) & 1\ (c_1) & -18\ (c_0) & \boxed{2\ (r)} \end{array}$$

$$q(x) = c_2x^2 + c_1x + c_0 = 2x^2 + x - 18, \qquad r = 2.$$

2. Find a, b, c, d satisfying $f(x) = a + b(x-2) + c(x-2)^2 + d(x-2)^3$. Show work.

   Solution. By synthetic division below, $a = r = 2, b = -8, c = 9, d = 2$.

$$\begin{array}{c|ccc} 2 & 2 & 1 & -18 \\ & & 4 & 10 \\ \hline 2 & 2 & 5 & \boxed{-8\ (b)} \\ & & 4 & \\ \hline & 2\ (d) & \boxed{9\ (c)} & \end{array}$$

   $2x^2 + x - 18 = (2x+5)(x-2) - 8$, $2x + 5 = 2(x-2) + 9$.

$$\begin{aligned} f(x) &= 2x^3 - 3x^2 - 20x + 38 = (2x^2 + x - 18)(x-2) + 2 \\ &= ((2x+5)(x-2) - 8)(x-2) + 2 = ((2(x-2)+9)(x-2) - 8)(x-2) + 2 \\ &= 2 - 8(x-2) + 9(x-2)^2 + 2(x-2)^3. \end{aligned}$$

3. Find a polynomial $Q(x)$ of degree 3 such that $Q(2) = 0$, $Q(3) = 1$, $Q(4) = 0$, $Q(5) = 0$.

   Solution.

$$Q(x) = \frac{(x-2)(x-4)(x-5)}{(3-2)(3-4)(3-5)} \left(= \frac{1}{2}(x-2)(x-4)(x-5) = \frac{1}{2}x^3 - \frac{11}{2}x^2 + 19x - 20 \right).$$

4. Let $h(x) = a(x-3)(x-4)(x-5) + b(x-2)(x-4)(x-5) + c(x-2)(x-3)(x-5) + d(x-2)(x-3)(x-4)$. Find a, b, c, d when $h(2) = 12$, $h(3) = 1$, $h(4) = -2$, $h(5) = 6$. Show work.

$$\begin{aligned} \text{Solution. } 12 &= h(2) = a(2-3)(2-4)(2-5) = -6a, \quad a = -2, \\ 1 &= h(3) = b(3-2)(3-4)(3-5) = 2b, \quad b = \frac{1}{2}, \\ -2 &= h(4) = c(4-2)(4-3)(4-5) = -2c, \quad c = 1 \\ 6 &= h(5) = d(5-2)(5-3)(5-4) = 6d, \quad d = 1 \end{aligned}$$

5. Find a polynomial of degree 7 satisfying $g(2) = 12$, $g(3) = 1$, $g(4) = -2$, $g(5) = 6$.

   Solution.

$$\begin{aligned} g(x) &= h(x) + x^3(x-2)(x-3)(x-4)(x-5) \\ &= x^3(x-2)(x-3)(x-4)(x-5) - 2(x-3)(x-4)(x-5) \\ &\quad + \frac{1}{2}(x-2)(x-4)(x-5) + (x-2)(x-3)(x-5) + (x-2)(x-3)(x-4). \end{aligned}$$

Quiz 5

ID#: **Name:**

1. Find the limits of the following. If there is no limit, or diverge, briefly explain why. Show work!

   (a) $\lim_{n\to\infty} \frac{10^n}{9^n}$

   (b) $\lim_{n\to\infty} \frac{1-n^2}{1+n^2}$

   (c) $\lim_{n\to\infty} \frac{2n-3n^2}{1-2n+2n^2-n^3}$

   (d) $\lim_{x\to 3} \frac{x^2+x-3}{x-3}$

   (e) $\lim_{x\to 3} \frac{x^2-2x-3}{x^2-9}$

   (f) $\lim_{x\to 3} \frac{x^2-x+3}{x^2+x-3}$

2. Find the limit of the following. Show work.
   $\lim_{x\to 3} \frac{-x^4+11x^3-41x^2+57x-18}{x^3-7x^2+15x-9}$

3. Let $f(x) = q(x)(x-3)+5$, where $q(x)$ is a polynomial. Suppose $f(0) = -2$. Explain that there is c $(0 < c < 3)$ such that $f(c) = 0$.

Message: ICU aims to nurture trustworthy global citizen. What do you think is the key? ICU は「信頼される地球市民を育む」ことを目指していますが、鍵は何だと思いますか。(If you don't want your message to be posted, write "Do Not Post."「ホームページ掲載不可」の場合は明記のこと。)

Solutions to Quiz 5

1. Find the limits of the following. If there is no limit, or diverge, briefly explain why. Show work!

   (a) $\displaystyle\lim_{n\to\infty}\frac{10^n}{9^n}=\lim_{n\to\infty}\left(\frac{10}{9}\right)^n$: Diverge (The limit does not exist.) $\quad \left|\frac{10}{9}\right|>1.$

   (b) $\displaystyle\lim_{n\to\infty}\frac{1-n^2}{1+n^2}=\lim_{n\to\infty}\frac{\frac{1}{n^2}-1}{\frac{1}{n^2}+1}=\frac{-1}{1}=-1.\qquad \lim_{n\to\infty}\frac{1}{n^2}=\left(\lim_{n\to\infty}\frac{1}{n}\right)^2=0.$

   (c) $\displaystyle\lim_{n\to\infty}\frac{2n-3n^2}{1-2n+2n^2-n^3}=\lim_{n\to\infty}\frac{2\frac{1}{n^2}-3\frac{1}{n}}{\frac{1}{n^3}-2\frac{1}{n^2}+2\frac{1}{n}-1}=0.$

   (d) $\displaystyle\lim_{x\to3}\frac{x^2+x-3}{x-3}=\lim_{x\to3}\frac{(x+4)(x-3)+9}{x-3}\sim\frac{9}{0}$: Divergent (The limit does not exist.)

   (e) $\displaystyle\lim_{x\to3}\frac{x^2-2x-3}{x^2-9}=\lim_{x\to3}\frac{((x-3)+4)(x-3)}{((x-3)+6)(x-3)}=\lim_{x\to3}\frac{(x-3)+4}{(x-3)+6}=\frac{4}{6}=\frac{2}{3}.$

   (f) $\displaystyle\lim_{x\to3}\frac{x^2-x+3}{x^2+x-3}=\lim_{x\to3}\frac{(x+2)(x-3)+9}{(x+4)(x-3)+9}=\frac{9}{9}=1.$

2. Find the limit of the following. Show work.

$$\lim_{x\to3}\frac{-x^4+11x^3-41x^2+57x-18}{x^3-7x^2+15x-9}$$

3	−1	11	−41	57	−18
		−3	24	−51	18
3	−1	8	−17	6	0
		−3	15	−6	
3	−1	5	−2	0	
		−3	6		
	−1	2	4		

3	1	−7	15	−9
		3	−12	9
3	1	−4	3	0
		3	−3	
3	1	−1	0	
		3		
	1	2		

$$\begin{aligned}&\lim_{x\to3}\frac{-x^4+11x^3-41x^2+57x-18}{x^3-7x^2+15x-9}\\ &=\lim_{x\to3}\frac{((-x+2)(x-3)+4)(x-3)^2}{((x-3)+2)(x-3)^2}=\lim_{x\to3}\frac{(-x+2)(x-3)+4}{(x-3)+2}=\frac{4}{2}=2.\end{aligned}$$

3. Let $f(x)=q(x)(x-3)+5$, where $q(x)$ is a polynomial. Suppose $f(0)=-2$. Explain that there is c $(0<c<3)$ such that $f(c)=0$.

   Solution. Since $f(x)$ is a polynomial, it is continuous. Since $f(x)=q(x)(x-3)+5$, $f(3)=5$. By assumption, $f(0)=-2$. Since 0 is in between 5 and -2, by Proposition 5.3 (Intermediate Value Theorem) there is a number c in the interval $[0,2]$ such that $f(c)=0$.

Quiz 6

ID#: **Name:**

1. Find the derivatives of the following functions $y = f(x)$.

   (a) $y = \frac{1}{5}x^5 - \frac{1}{x^5} + 5\sqrt[5]{x}$, $(x > 0)$.

   (b) $y = \frac{e^x - 1}{e^x + 1}$

   (c) $y = (x^2 + 1)^{50}$

   (d) $y = x \log x - x$, $(x > 0)$.

2. Find the limit of the following.
   $\lim_{x \to 0} \frac{e^x - 1 - x}{x^2 + x^3 + x^4}$.

3. Suppose $f(x)$ is differentiable, $g(x) = f'(x) = x^3 - 4x^2 + 5x - 2$ and $f(0) = 2$.

   (a) Find the equation of the tangent line to the curve $y = f(x)$ at $x = 0$.

   (b) Find the derivative $h(x)$ of $g(x)$, and the derivative of $h(x)$, i.e., find $h(x) = g'(x) = f''(x)$, and $h'(x) = f'''(x)$.

   (c) Determine whether $f(x)$ is increasing, decreasing, a local maximum or a local minimum at $x = 2$. State your reason.

   (d) Determine whether $f(x)$ is increasing, decreasing, a local maximum or a local minimum at $x = 1$. State your reason.

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Solutions to Quiz 6

1. Find the derivatives of the following functions $y = f(x)$.

   (a) $y = \dfrac{1}{5}x^5 - \dfrac{1}{x^5} + 5\sqrt[5]{x} = \dfrac{1}{5}x^5 - x^{-5} + 5x^{\frac{1}{5}}$, $(x > 0)$.

   $$y' = x^4 + 5x^{-6} + x^{-\frac{4}{5}} = x^4 + \frac{5}{x^6} + \frac{1}{\sqrt[5]{x^4}}. \qquad ((x^a)' = ax^{a-1})$$

   (b) $y = \dfrac{e^x - 1}{e^x + 1}$, $\quad y' = \dfrac{e^x(e^x + 1) - (e^x - 1)e^x}{(e^x + 1)^2} = \dfrac{2e^x}{(e^x + 1)^2}$.

   (c) $y = (x^2 + 1)^{50}$, $\quad y' = 50(x^2 + 1)^{49}(x^2 + 1)' = 50(x^2 + 1)^{49}(2x) = 100x(x^2 + 1)^{49}$.

   (d) $y = x \log x - x$, $(x > 0)$. $\quad y' = \log x + x \cdot \dfrac{1}{x} - 1 = \log x$. $\quad \left((\log x)' = \dfrac{1}{x}\right)$.

2. Find the limit of the following.
   $$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2 + x^3 + x^4} = \lim_{x \to 0} \frac{e^x - 1}{2x + 3x^2 + 4x^3} = \lim_{x \to 0} \frac{e^x}{2 + 6x + 12x^2} = \frac{1}{2}.$$

3. Suppose $f(x)$ is differentiable, $g(x) = f'(x) = x^3 - 4x^2 + 5x - 2$ and $f(0) = 2$.

   (a) Find the equation of the tangent line to the curve $y = f(x)$ at $x = 0$.
   Solution. Since $f(0) = 2$ and $f'(0) = -2$, the equation of the tangent line is:
   $$y = f(0) + f'(0)(x - 0) = 2 - 2x = -2(x - 1).$$

   (b) Find the derivative $h(x)$ of $g(x)$, and the derivative of $h(x)$, i.e., find $h(x) = g'(x) = f''(x)$, and $h'(x) = f'''(x)$.
   Solution.
   $$f''(x) = h(x) = 3x^2 - 8x + 5, \quad f'''(x) = h'(x) = 6x - 8.$$

   (c) Determine whether $f(x)$ is increasing, decreasing, a local maximum or a local minimum at $x = 2$. State your reason.
   Solution. By the calculation below using synthetic division, $f'(2) = 0$ and $f''(2) = 1 > 0$. Hence by the second derivative test, $f(x)$ is a local minimum at $x = 2$.

   $$\begin{array}{c|cccc} 2 & 1 & -4 & 5 & -2 \\ & & 2 & -4 & 2 \\ \hline & 1 & -2 & 1 & \boxed{0} \end{array} \qquad \begin{array}{c|ccc} 2 & 3 & -8 & 5 \\ & & 6 & -4 \\ \hline & 3 & -2 & \boxed{1} \end{array}$$

   (d) Determine whether $f(x)$ is increasing, decreasing, a local maximum or a local minimum at $x = 1$. State your reason.
   Solution. By the calculation below using synthetic division, $f'(1) = f''(1) = 0$ and $f'''(1) = 6 - 8 = -2 < 0$. Hence by the second derivative test, $f'(x)$ has a local maximum at $x = 1$ with $f'(1) = 0$. Hence $f'(x) < 0$ near $x = 1$ and $f(x)$ is decreasing.

   $$\begin{array}{c|cccc} 1 & 1 & -4 & 5 & -2 \\ & & 1 & -3 & 2 \\ \hline & 1 & -3 & 2 & \boxed{0} \end{array} \qquad \begin{array}{c|ccc} 1 & 3 & -8 & 5 \\ & & 3 & -5 \\ \hline & 3 & -5 & \boxed{0} \end{array}$$

   Another Solution. Since $f'''(1) < 0$, $f''(x)$ is decreasing near $x = 1$ and $f''(1) = 0$. Hence if x is near 1 and $x < 1$, $f''(x) > 0$. Similarly, if x is near 1 and $x > 1$, $f''(x) < 0$. Therefore, $f'(x)$ is increasing when x is near 1 and $x < 1$ and decreasing when x is near 1 and $x > 1$. Since $f'(1) = 0$, $f'(x) < 0$ when x is near 1 and $f(x)$ is decreasing

Quiz 7

ID#: **Name:**

1. Find the derivative of $f(x) = e^{-x^2}$.

2. Find $F'(x)$ when $F(x) = \int_0^x te^{-t^2} dt$.

3. Evaluate the definite integral $\int_0^1 xe^{-x^2} dx$.

4. Compute the following.

   (a) $\int (3x^5 - 2x^3 + x^2 - 1) dx$

   (b) $\int (\frac{12}{x^7} + \frac{1}{2}\sqrt{x}) dx$

   (c) $\int (\frac{1}{2x} + e^{2x}) dx$

   (d) $\int (5x + 1)^7 dx$

   (e) $\int_{-1}^1 (t + 1)^3 dt$

5. Let $y = f(x)$. Solve the following differential equations.

   (a) $y' = \frac{x}{10}$, $y(0) = 10$.

   (b) $y' = \frac{y}{10}$, $y(0) = 10$.

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Solutions to Quiz 7

1. Find the derivative of $f(x) = e^{-x^2}$.

   Solution. By Chain Rule,

$$f'(x) = (e^{-x^2})' = e^{-x^2}(-x^2)' = -2xe^{-x^2}.$$

2. Find $F'(x)$ when $F(x) = \int_0^x te^{-t^2}dt$.

   Solution. By FTC,

$$F'(x) = xe^{-x^2}.$$

3. Evaluate the definite integral $\int_0^1 xe^{-x^2}dx$.

   Solution. Since $(e^{-x^2})' = -2xe^{-x^2}$,

$$\int_0^1 xe^{-x^2}dx = -\frac{1}{2}\int_0^1 -2xe^{-x^2}dx = -\frac{1}{2}\left[e^{-x^2}\right]_0^1 = -\frac{1}{2}(e^{-1}-1) = \frac{1}{2}(1-\frac{1}{e}).$$

4. Compute the following.

   (a) $$\int(3x^5 - 2x^3 + x^2 - 1)dx = \frac{3}{5+1}x^{5+1} - \frac{2}{3+1}x^{3+1} + \frac{1}{2+1}x^3 - x + C$$
$$= \frac{1}{2}x^6 - \frac{1}{2}x^4 + \frac{1}{3}x^3 - x + C.$$

   (b) $$\int(\frac{12}{x^7} + \frac{1}{2}\sqrt{x})dx = \int(12x^{-7} + \frac{1}{2}x^{1/2})dx$$
$$= \frac{12}{-7+1}x^{-7+1} + \frac{1}{2(\frac{1}{2}+1)}x^{\frac{1}{2}+1} + C = -2x^{-6} + \frac{1}{3}x^{\frac{3}{2}} + C = -\frac{2}{x^6} + \frac{1}{3}(\sqrt{x})^3 + C.$$

   (c) $\int(\frac{1}{2x} + e^{2x})dx = \frac{1}{2}\log|x| + \frac{1}{2}e^{2x} + C.$

   (d) $\int(5x+1)^7dx = \frac{1}{40}\int 40(5x+1)^7dx = \frac{1}{40}(5x+1)^8 + C.$

   (e) $\int_{-1}^1(t+1)^3dt = \left[\frac{1}{4}(t+1)^4\right]_{-1}^1 = \frac{1}{4}2^4 = 4.$

5. Let $y = f(x)$. Solve the following differential equations.

   (a) $y' = \frac{x}{10}$, $y(0) = 10$.

$$y = \frac{1}{20}x^2 + C,\ 10 = y(0) = C. \text{ Hence, } y = \frac{1}{20}x^2 + 10.$$

   (b) $y' = \frac{y}{10}$, $y(0) = 10$.

$$\frac{dy}{dx} = \frac{y}{10}. \text{ Hence, } \int\frac{1}{y}dy = \int\frac{1}{10}dx$$

   Therefore,

$$\log|y| = \frac{1}{10}x + C, \quad y = e^Ce^{\frac{x}{10}}.$$

   Since $y(0) = 10$, $e^C = 10$.

$$y = 10e^{\frac{x}{10}}.$$