2 Linear Transformations from $\boldsymbol{R}^n$ to $\boldsymbol{R}^m$

Definition 2.1 Let X and Y be sets. A *function* (or *mapping*) f is a rule that associates with each element $a \in X$ one and only element $b \in Y$.

- $f : X \to Y$ $(a \mapsto b = f(a))$.
- b is the *image* of a under f, or $f(a)$ is the value of f at a.
- X is the *domain* of f and Y is the *codomain* of f.
- $\mathrm{Im} f = \{f(a) \mid a \in X\}$ is called the *range* of f.

Two functions (mappings) $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ are equal if $X_1 = X_2$, $Y_1 = Y_2$ and $f_1(a) = f_2(a)$ for all $a \in X_1 = X_2$.

Definition 2.2 If the domain of a function T is $\boldsymbol{R}^n$ and the codomaini R^m then T is called a transformation from $\boldsymbol{R}^n$ to $\boldsymbol{R}^m$.

A mapping $T : \boldsymbol{R}^n \to \boldsymbol{R}^m$ is called a *linear transformation* if

$$T : \boldsymbol{R}^n \to \boldsymbol{R}^m \ (\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \mapsto T\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \\ \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \end{bmatrix})$$

Let

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \ A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ & & \cdots & \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}$$

Then the linear transformation can be written as

$$T : \boldsymbol{R}^n \to \boldsymbol{R}^m \ (\boldsymbol{x} \mapsto A\boldsymbol{x}).$$

The matrix $A = [a_{i,j}]$ is called the *standard matrix* of T and write $A = [T]$.

Conversely if A is an $m \times n$ matrix and the mapping from R^n to R^m is defined by $\boldsymbol{x} \mapsto A\boldsymbol{x}$, then the linear transformation is denoted by T_A. In particular $[T_A] = A$.

Theorem 2.1 *Let $T_1 : \boldsymbol{R}^n \to \boldsymbol{R}^m$ and $T_2 : \boldsymbol{R}^m \to \boldsymbol{R}^\ell$ be linear transformations. Then the composition of T_2 with T_1 defined by*

$$T_2 \circ T_1 : \boldsymbol{R}^n \to \boldsymbol{R}^\ell \ (\boldsymbol{x} \mapsto T_2(T_1(\boldsymbol{x}))).$$

is a linear transformation and $[T_2 \circ T_1] = [T_2][T_1]$.

Theorem 2.2 (4.3.2) *A transformation $T : \boldsymbol{R}^n \to \boldsymbol{R}^m$ is linear if and only if the following hold for all $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{R}^n$ and for every scalar c.*

(a) $T(\boldsymbol{u} + \boldsymbol{v}) = T(\boldsymbol{u}) + T(\boldsymbol{v})$ (b) $T(c\boldsymbol{u}) = cT(\boldsymbol{u})$.

Proof. Only if part is clear.

Suppose T satisfies (a) and (b). Then by induction it is easy to show that

$$T(\boldsymbol{u}_1 + \boldsymbol{u}_2 + \cdots + \boldsymbol{u}_t) = T(\boldsymbol{u}_1) + T(\boldsymbol{u}_2) + \cdots + T(\boldsymbol{u}_t).$$

Let $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$ be uinit vectors in $\boldsymbol{R}^n$, and $A = [T(\boldsymbol{e}_1), T(\boldsymbol{e}_2), \ldots, T(\boldsymbol{e}_n)]$. If $\boldsymbol{x} = [x_1, x_2, \ldots, x_n]^T$, then

$$\begin{aligned} A\boldsymbol{x} &= x_1 T(\boldsymbol{e}_1) + x_2 T(\boldsymbol{e}_2) + \cdots + x_n T(\boldsymbol{e}_n) \\ &= T(x_1\boldsymbol{e}_1) + T(x_2\boldsymbol{e}_2) + \cdots + T(x_n\boldsymbol{e}_n) \\ &= T(x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + \cdots + x_n\boldsymbol{e}_n) \\ &= T(\boldsymbol{x}) \end{aligned}$$

This proves that T is a linear transformation. ■

Corollary 2.3 (4.3.3) *If T is a linear transformation from $\boldsymbol{R}^n$ to $\boldsymbol{R}^m$ and $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$, then*

$$[T] = [T\boldsymbol{e}_1, T\boldsymbol{e}_2, \ldots, T\boldsymbol{e}_n].$$

Example 2.1 Let $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$ be unit vectors in $\boldsymbol{R}^n$ and let $m < n$. Let

$$T : \boldsymbol{R}^n \to \boldsymbol{R}^n \ (\boldsymbol{x} \mapsto (\boldsymbol{x} \cdot \boldsymbol{e}_1)\boldsymbol{e}_1 + (\boldsymbol{x} \cdot \boldsymbol{e}_2)\boldsymbol{e}_2 + \cdots + (\boldsymbol{x}_m \cdot \boldsymbol{e}_m)\boldsymbol{e}_m).$$

Then T is a linear transformation (operator), which is called a *projection*.

Definition 2.3 Let $f : X \to Y$ be a function (or mapping).

(a) If $\mathrm{Im}(f) = f(X) = Y$, then f is said to be surjective or onto.

(b) If $f(a) \neq f(a')$ whenever $a \neq a'$, f is said to be injective or one-to-one. f is injective iff $f(a) = f(a')$ implies $a = a'$ for all $a, a' \in X$.

(c) If f is one-to-one and onto, f is said to be bijective.

Recall the following.

Theorem 2.4 (2.3.6) *If A is an $n \times n$ matrix, then the following statements are equivalent.*

(a) *A is invertible.*

(b) *$A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution.*

(c) *The reduced echelon form of A is I_n.*

(d) *A can be expressed as a product of elementary matrices.*

(e) *$A\boldsymbol{x} = \boldsymbol{b}$ is consistent for every $n \times 1$ matrix $\boldsymbol{b}$.*

(f) *$A\boldsymbol{x} = \boldsymbol{b}$ has exactly one solution for every $n \times 1$ matrix $\boldsymbol{b}$.*

(g) $\det(A) \neq 0$.

Theorem 2.5 (4.3.1) *If A is an $n \times n$ matrix and $T_A : \boldsymbol{R}^n \to \boldsymbol{R}^n$ is multiplication by A, then the following statements are equivalent.*

(a) *A is invertible.*

(b) *T_A is surjective.*

(c) *T_A is injective.*

(d) *T_A is bijective.*

Remarks. T_A is injective if and only if $T_A(\boldsymbol{x}) = \boldsymbol{0}$ implies $\boldsymbol{x} = \boldsymbol{0}$.

Exercise 2.1 [Quiz 2] For $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)^T$ be a nonzero vector in $\boldsymbol{R}^n$, Let

$$\tau_{\boldsymbol{u}} : \boldsymbol{R}^n \to \boldsymbol{R}^n \ (\boldsymbol{x} \mapsto \boldsymbol{x} - \frac{2\boldsymbol{x} \cdot \boldsymbol{u}}{\|\boldsymbol{u}\|^2}\boldsymbol{u}).$$

1. Show that $\tau_{\boldsymbol{u}}$ is a linear transformation.

2. Let $\boldsymbol{v} = (1, -1, 0, \ldots, 0)^T$. Find the standard matrix $[\tau_{\boldsymbol{v}}]$.

3. Suppose T is a linear transformation from $\boldsymbol{R}^n$ to $\boldsymbol{R}^n$ such that $T(\boldsymbol{u}) = -\boldsymbol{u}$, $T(\boldsymbol{w}) = \boldsymbol{w}$ whenever $\boldsymbol{w} \cdot \boldsymbol{u} = 0$. Show that $T = \tau_{\boldsymbol{u}}$. (Hint: If $\alpha = \frac{\boldsymbol{x} \cdot \boldsymbol{u}}{\|\boldsymbol{u}\|^2}$, $(\boldsymbol{x} - \alpha\boldsymbol{u}) \cdot \boldsymbol{u} = 0$.)

課題

A. 11 章の一つの節を選び、その内容、またはそれに関連するトピックから線形代数の応用例で興味を持ったものを選び、Linear Algebra II 受講生に分かり易いように解説し、 A4 2 ページにまとめよ。Power Point などの Presentation Tool で作成する場合は、16 スライドが最大。A4 1 ページに 8 スライドを印刷しても読める程度にして下さい。

B. 上で選んだ節の問題を 5 問以上解答し、A4 の紙（枚数問わず）に書いて提出。

- 提出期限は 1 月 30 日 1 時 50 分まで。提出場所は、Take-Home Quiz 提出場所と同じ。

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- A, B を期限までに提出すれば 10 点、その質により、のこりの点数 10 点を与えます。満点は 20 点。