8 Applications of Determinants

8.1 Cramer's Rule

Theorem 8.1 (Theorem 7 (Cramer's Rule in page 195)) *If $A\boldsymbol{x} = \boldsymbol{b}$ is a system of n linear equations in n unknowns such that $\det(A) \neq 0$, then the system has a unique solution. The solution is*

$$x_1 = \frac{\det(A_1(\boldsymbol{b}))}{\det(A)},\ x_2 = \frac{\det(A_2(\boldsymbol{b}))}{\det(A)}, \ldots,\ x_n = \frac{\det(A_n(\boldsymbol{b}))}{\det(A)},$$

where $A_j(\boldsymbol{b})$ is the matrix obtained by replacing the jth column of A by $\boldsymbol{b}$.

Proof. By Review 2 (c), for $j = 1, 2, \ldots, n$,observe the following.

$$\begin{aligned} \det(A) &= a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}. \\ \det(A_j(\boldsymbol{b})) &= b_1C_{1j} + b_2C_{2j} + \cdots + b_nC_{nj}. \end{aligned}$$

Since $A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A) = \frac{1}{\det(A)}\tilde{A}^{\top}$,

$$\begin{aligned} \boldsymbol{x} &= A^{-1}\boldsymbol{b} = \frac{1}{\det(A)}\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \\ &= \frac{1}{\det(A)}\begin{bmatrix} b_1C_{11} + b_2C_{21} + \cdots + b_nC_{n1} \\ b_1C_{12} + b_2C_{22} + \cdots + b_nC_{n2} \\ \cdots \\ b_1C_{1n} + b_2C_{2n} + \cdots + b_nC_{nn} \end{bmatrix} = \frac{1}{\det(A)}\begin{bmatrix} \det(A_1(\boldsymbol{b})) \\ \det(A_2(\boldsymbol{b})) \\ \vdots \\ \det(A_n(\boldsymbol{b})) \end{bmatrix}. \end{aligned}$$

The last equality follows from our observation above. Hence we have the formula. ∎

Example 8.1

$$\left\{ \begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 &= b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 &= b_2 \end{aligned} \right.,\ \det\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} = a_{1.1}a_{2,2} - a_{2,1}a_{1,2} \neq 0$$

$$\Rightarrow x_1 = \frac{\det\begin{bmatrix} b_1 & a_{1,2} \\ b_2 & a_{2,2} \end{bmatrix}}{\det\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}} = \frac{a_{2,2}b_1 - a_{1,2}b_2}{a_{1.1}a_{2,2} - a_{2,1}a_{1,2}},\ x_2 = \frac{a_{1,1}b_2 - a_{2,1}b_1}{a_{1.1}a_{2,2} - a_{2,1}a_{1,2}}.$$

$$\left\{ \begin{aligned} 3x_1 - 5x_2 &= 4 \\ 2x_1 + 7x_2 &= -3 \end{aligned} \right. \Rightarrow x_1 = \frac{\begin{vmatrix} 4 & -5 \\ -3 & 7 \end{vmatrix}}{\begin{vmatrix} 3 & -5 \\ 2 & 7 \end{vmatrix}} = \frac{13}{31},\ x_2 = \frac{\begin{vmatrix} 3 & 4 \\ 2 & -3 \end{vmatrix}}{\begin{vmatrix} 3 & -5 \\ 2 & 7 \end{vmatrix}} = \frac{-17}{31}.$$

8.2 Combinatorial Definition of Determinants

Definition 8.1 A *permutation*（置換） of the set of integers（整数） $\{1, 2, \ldots, n\}$ is an arrangement（並び替え） of these integers in some order without omissions or repetitions. Let S_n denote the set of all permutations of $\{1, 2, \ldots, n\}$. Let $\sigma = (i_1, i_2, \ldots, i_n)$ be a permutation. Then the number of inversions（転置数）, denoted by $\ell(\sigma)$, is defined by

$$\ell(\sigma) = |\{(j,k) \mid j < k,\ i_j > i_k\}|, \text{ and } \mathrm{sign}(\sigma) = (-1)^{\ell(\sigma)}$$

is called the *signature*（符号） of σ.

Let $A = (a_{i,j})$ be a square matrix of size n. Then

$$\begin{aligned}\det(A) &= \sum_{(i_1,i_2,\ldots,i_n)\in S_n} \mathrm{sign}((i_1, i_2, \ldots, i_n)) a_{1,i_1} a_{2,i_2} \cdots a_{n,i_n} \\ &= \sum_{(i_1,i_2,\ldots,i_n)\in S_n} (-1)^{\ell((i_1,i_2,\ldots,i_n))} a_{1,i_1} a_{2,i_2} \cdots a_{n,i_n}.\end{aligned}$$

8.3 Vector Product in $\mathbb{R}^3$

Let $\boldsymbol{u}, \boldsymbol{v}$ be (column) vectors in $\mathbb{R}^3$. Then the *vector product*（ベクトル積） (or *cross product*) of $\boldsymbol{u}$ and $\boldsymbol{v}$ is:

$$\begin{aligned}\boldsymbol{u} \times \boldsymbol{v} &= [u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1]^\top \\ &= \left[\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, \begin{vmatrix} u_3 & u_1 \\ v_3 & v_1 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\right]^\top = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.\end{aligned}$$

Let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ be vectors in $\mathbb{R}^3$. Then the *scalar triple product* of $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ is defined as follows.

$$\begin{aligned}\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) &= [u_1, u_2, u_3]^\top \cdot \left[\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}, -\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}, \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}\right]^\top \\ &= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \\ &= \boldsymbol{v} \cdot (\boldsymbol{w} \times \boldsymbol{u}) = \boldsymbol{w} \cdot (\boldsymbol{u} \times \boldsymbol{v})\end{aligned}$$

The following hold:

$$\boldsymbol{u} \times \boldsymbol{v} = -\boldsymbol{v} \times \boldsymbol{u},\ \boldsymbol{u} \cdot (\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{v} \cdot (\boldsymbol{u} \times \boldsymbol{v}) = 0.$$

Theorem 8.2 (Determinants as Area or Volume in page 198) *If A is a 2×2 matrix, the area of the parallelogram*（平行四辺形） *determined by the columns of A is $|\det(A)|$. If A is a 3×3 matrix, the volume of the parallelepiped*（平行六面体） *determined by the columns of A is $|\det(A)|$.*