7 Determinants

7.1 Definition of Determinants and Cofactor Expansions

Matrices of Size Two

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \; B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}, \; AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} ax+bz & ay+bw \\ cx+dz & cy+dw \end{bmatrix}.$$

Define that $\det(A) = ad - bc$.

Claim 1. $\det(AB) = \det(A)\det(B)$.

$$\begin{aligned} \det(AB) &= (ax+bz)(cy+dw) - (ay+bw)(cx+dz) \\ &= acxy + adxw + bcyz + bdzw - acxy - adyz - bcxw - bdzw \\ &= (ad-bc)xw - (ad-bc)yz = (ad-bc)(xw-yz) \\ &= \det(A)\det(B). \end{aligned}$$

Recall that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = (ad-bc) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Claim 2. We have $\det(A) = ad - bc \neq 0 \Leftrightarrow A$ is invertible.

Proof. If $\det(A) = ad - bc \neq 0$, then A is invertible because

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}.$$

If A is invertible, then there is a matrix $AB = I$. By Claim 1,

$$1 = \det(I) = \det(AB) = \det(A)\det(B).$$

Therefore $\det(A) \neq 0$. ∎

Definition 7.1 (page 183) Let $A = [a_{ij}]$ be an $n \times n$ matrix. We define the *determinant* (行列式) of A denoted by $\det(A)$ (or $|A|$) recursively as follows.

1. If $n = 1$ and $A = [a]$, then $\det(A) = a$.

2. Suppose $n > 1$ and the determinants of all $(n-1) \times (n-1)$ matrices are defined. Then for $1 \leq i, j \leq n$, A_{ij} denotes the $(n-1) \times (n-1)$ submatrix formed by deleting the ith row and jth column of A. The (i,j)-*cofactor* (余因子) of A is the number $C_{ij} = (-1)^{i+j}\det(A_{ij})$. Let

$$\begin{aligned} \det(A) &= a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + \cdots + (-1)^{1+n} a_{1n}\det(A_{1n}) \\ &= a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}. \end{aligned}$$

Matrices of Size Three: Formula of Sarras

$$\begin{aligned}\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} &= a_{1,1}\begin{vmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{vmatrix} - a_{1,2}\begin{vmatrix} a_{2,1} & a_{2,3} \\ a_{3,1} & a_{3,3} \end{vmatrix} + a_{1,3}\begin{vmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{vmatrix} \\ &= a_{1,1}(a_{2,2}a_{3,3} - a_{2,3}a_{3,2}) - a_{1,2}(a_{2,1}a_{3,3} - a_{2,3}a_{3,1}) + a_{1,3}(a_{2,1}a_{3,2} - a_{2,2}a_{3,1}) \\ &= a_{1,1}a_{2,2}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} + a_{1,3}a_{2,1}a_{3,2} - a_{1,1}a_{2,3}a_{3,2} - a_{1,2}a_{2,1}a_{3,3} - a_{1,3}a_{2,2}a_{3,1}.\end{aligned}$$

Definition 7.2 (page 197) Let $A = [a_{ij}]$ be an $n \times n$ matrix. Then the left matrix is called the *matrix of cofactors*（余因子行列） of A, and the matrix on the right that is the transpose of the left is called the *adjugate*（随伴行列） (or *classical adjoint*) of A and denoted by $\operatorname{adj}(A)$.

$$\tilde{A} = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}, \ \operatorname{adj}(A) = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}.$$

Theorem 7.1 (Theorem 1 in page 184 and Theorem 8 in page 197)
Let $A = [a_{ij}]$ be an $n \times n$ matrix and $\operatorname{adj}(A)$ the adjugate of A. Then

$$A \cdot \operatorname{adj}(A) = \det(A)I = \operatorname{adj}(A) \cdot A.$$

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i,1} & a_{i,2} & \cdots & a_{i,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix} \begin{bmatrix} C_{1,1} & \cdots & C_{j,1} & \cdots & C_{n,1} \\ C_{1,2} & \cdots & C_{j,2} & \cdots & C_{n,2} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ C_{1,n} & \cdots & C_{j,n} & \cdots & C_{n,n} \end{bmatrix} = \begin{bmatrix} \det(A) & 0 & \cdots & 0 \\ 0 & \det(A) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \det(A) \end{bmatrix}$$

Corollary 7.2 (Theorem 1 in page 184 and Theorem 8 in page 197)
Let $A = [a_{ij}]$ be an $n \times n$ matrix and C_{ij} the (i, j)-cofactor of A. Then the following hold.

(i) $\det(A) = a_{i,1}C_{i,1} + a_{i,2}C_{i,2} + \cdots + a_{i,n}C_{i,n}$ *for* $i = 1, 2, \ldots, n$.

(ii) $\det(A) = a_{1,j}C_{1,j} + a_{2,j}C_{2,j} + \cdots + a_{n,j}C_{n,j}$ *for* $j = 1, 2, \ldots, n$.

(iii) $a_{i,1}C_{j,1} + a_{i,2}C_{j,2} + \cdots + a_{i,n}C_{j,n} = 0$ *for* $i, j = 1, 2, \ldots, n$ *with* $i \neq j$.

(iv) $a_{1,j}C_{1,i} + a_{2,j}C_{2,i} + \cdots + a_{n,j}C_{i,n} = 0$ *for* $i, j = 1, 2, \ldots, n$ *with* $i \neq j$.

(v) *A is invertible if and only if* $\det(A) \neq 0$. *In this case*

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \frac{1}{\det(A)} \begin{bmatrix} C_{1,1} & C_{2,1} & \cdots & C_{n,1} \\ C_{1,2} & C_{2,2} & \cdots & C_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1,n} & C_{2,n} & \cdots & C_{n,n} \end{bmatrix}.$$

Example 7.1 Let $n = 2$.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \; A_{1,1} = d, A_{1,2} = c, A_{2,1} = b, A_{2,2} = a.$$

$$\tilde{A} = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}, \mathrm{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \text{ and } \det(A) = ad - bc.$$

Example 7.2 Let $n = 3$.

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 2 \\ 3 & 2 & 1 \end{bmatrix}, \; A^{-1} = \begin{bmatrix} -2 & 1 & 0 \\ 5 & -3 & 1 \\ -4 & 3 & -1 \end{bmatrix}.$$

Then

$$\tilde{A} = \begin{bmatrix} 2-4 & -(1-6) & 2-6 \\ -(1-2) & -3 & -(-3) \\ 2-2 & -(-1) & -1 \end{bmatrix} = \begin{bmatrix} -2 & 5 & -4 \\ 1 & -3 & 3 \\ 0 & 1 & -1 \end{bmatrix}, \mathrm{adj}(A) = \begin{bmatrix} -2 & 1 & 0 \\ 5 & -3 & 1 \\ -4 & 3 & -1 \end{bmatrix},$$

and $\det(A) = 0(-2) + 1 \cdot 5 + 1 \cdot (-4) = 1$.

$$A \cdot \mathrm{adj}(A) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 2 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 & 0 \\ 5 & -3 & 1 \\ -4 & 3 & -1 \end{bmatrix} = \det(A) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

7.2 Evaluation of Determinants

Definition 7.3 Let $A = (a_{i,j})$ be a square matrix of size n.

1. A is said to be an *upper triangular matrix*（上半三角行列） if $a_{i,j} = 0$ for all $i > j$.
2. A is said to be a *lower triangular matrix*（下半三角行列） if $a_{i,j} = 0$ for all $i < j$.
3. A is said to be a *diagonal matrix*（対角行列） if $a_{i,j} = 0$ for all $i \neq j$.

Theorem 7.3 (Theorems 2, 3, 5 in pages 185, 187 and 190)
Let $A = [a_{i,j}]$ be an $n \times n$ matrix.

(i) $\det(A) = \det(A^\top)$.

(ii) *If A is an upper triangular matrix, i.e., if $a_{i,j} = 0$ for all $i > j$, then* $\det(A) = a_{11}a_{22}\cdots a_{nn}$.

(iii) *The value of the determinant changes as follows by elementary row operations.*

(a) $A \xrightarrow{[i;c]} B \Rightarrow \det(B) = c\det(A)$, *and* $|E(i;c)A| = |E(i;c)||A| = c|A|$.

(b) $A \xrightarrow{[i,j]} B \Rightarrow \det(B) = -\det(A)$, *and* $|E(i,j)A| = |E(i,j)||A| = -|A|$.

(c) $A \xrightarrow{[i,j;c]} B \Rightarrow \det(B) = \det(A)$, *and* $|E(i,j;c)A| = |E(i,j;c)||A| = |A|$.

Similar results hold for elementary column operations（基本列変形） by (i).

Theorem 7.4 (Theorem 6 in page 191) *Let A and B be $n \times n$ matrices. Then*

$$\det(AB) = \det(A)\det(B).$$

Proof. If A is not invertible, then AB is not invertible. Hence $\det(AB) = 0 = \det(A)\det(B)$ by Corollary 7.2. On the other hand, if A is invertible, A is a product of elementary matrices. Let $A = E_1 E_2 \cdots E_\ell$. Now by consecutive applications of Theorem 7.3,

$$|AB| = |E_1 E_2 \cdots E_\ell B| = |E_1||E_2 \cdots E_\ell B| = |E_1||E_2| \cdots |E_\ell||B| = |E_1 E_2 \cdots E_\ell||B| = |A||B|.$$

This proves the assertion. ■

Example 7.3

$$\begin{vmatrix} 0 & 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 1 & 0 \\ 2 & 2 & 1 & 1 & 1 \\ 2 & -1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 \end{vmatrix} = 6, \quad \begin{vmatrix} -2 & 1 & 3 & 4 \\ 2 & 0 & -5 & 1 \\ 1 & 4 & 2 & -7 \\ 2 & -4 & -9 & 3 \end{vmatrix} = 110, \quad \begin{vmatrix} 0 & 1 & 0 \\ 6 & 1 & 3 \\ 0 & 4 & 3 \end{vmatrix} = -18.$$

Example 7.4

$$\begin{vmatrix} a & b & b & \cdots & b \\ b & a & b & \cdots & b \\ b & b & a & \cdots & b \\ & \cdots & \cdots & \cdots & \\ b & b & b & \cdots & a \end{vmatrix} = (a + (n-1)b)(a-b)^{n-1}.$$

Example 7.5 The following is called the Vandermonde's determinant.

$$\begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \cdots & x_3^{n-1} \\ & \cdots & \cdots & \cdots & \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{i>j}(x_i - x_j) = \prod_{j=1}^{n-1}\prod_{i=j+1}^{n}(x_i - x_j).$$

Example 7.6

$$\begin{vmatrix} x & 0 & 0 & \cdots & 0 & c_0 \\ -1 & x & 0 & \cdots & 0 & c_1 \\ 0 & -1 & x & \cdots & 0 & c_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & x + c_{n-1} \end{vmatrix} = c_0 + c_1 x + \cdots + c_{n-1}x^{n-1} + x^n.$$