6 Characterization of Inverse Matrices

6.1 Inverse of a Matrix

1. The identity matrix I of size n satisfies $AI = A = IA$ for all $n \times n$ matrix A.

2. Let A be a square matrix. The inverse of A is a matrix B such that $AB = I = BA$. The inverse is unique and we write $B = A^{-1}$. If there is an inverse A^{-1}, A is said to be invertible.

3. If A and B are invertible matrices of size n, then so is AB and $(AB)^{-1} = B^{-1}A^{-1}$. Moreover, if $A_1, A_2, \ldots, A_m$ are invertible matrices of size n, then their product $A_1A_2 \cdots A_m$ is also invertible and
$$(A_1A_2 \cdots A_m)^{-1} = A_m^{-1} \cdots A_2^{-1}A_1^{-1}.$$

4. For each elementary operation $[i;c]$, $[i,j]$, $[i,j;c]$, there is a corresponding elementary matrix E, denoted by $E(i;c)$, $E(i,j)$, $E(i,j;c)$ such that EA is exactly the one obtained by performing the corresponding elementary row operation to A. Moreover E is obtained from I by performing the corresponding elementary row operation.
$$[i;c] \Leftrightarrow E(i;c),\ [i,j] \Leftrightarrow E(i,j),\ [i.j;c] \Leftrightarrow E(i.j;c).$$

5. Elementary matrices are invertible:
$$E(i;c)^{-1} = E(i;\frac{1}{c}),\ E(i,j)^{-1} = E(i,j),\ E(i,j;c)^{-1} = E(i,j;-c).$$

6. Suppose $[A,I] \longrightarrow [I,B]$ by performing elementary row operations. Let $E_1, E_2, \ldots, E_m$ be corresponding elementary matrices. Then $B = A^{-1}$ and B and A can be expressed as a product of elementary matrices.
$$\begin{aligned}
[A,I] \to [I,B] &\Rightarrow E_mE_{m-1} \cdots E_2E_1[A,I] = [I,B] \\
&\Rightarrow [E_mE_{m-1} \cdots E_2E_1A, E_mE_{m-1} \cdots E_2E_1I] = [I,B] \\
&\Rightarrow B = E_mE_{m-1} \cdots E_1,\ BA = I \text{ and } B \text{ is invertible.} \\
&\Rightarrow A = B^{-1} = E_1^{-1}E_2^{-1} \cdots E_m^{-1} \text{ and } B = A^{-1}
\end{aligned}$$

7. If the reduced echelon form of $[A,I]$ is not of the form $[I,B]$, say $[D,B]$, then the last row of D is zero. Since $BA = D$ and D is not invertible, A is not invertible. Note that D is not invertible because the fact that the last row of D is zero implies the last row of DF is zero, and DF cannot be equal to I.

6.2 The Invertible Matrix Theorem

Theorem 6.1 (The Invertible Matrix Theorem (Theorem 8 in page 130)) *Let A be an $n \times n$ matrix. Then the following are equivalent.*

(a) *A is an invertible matrix.*

(b) *A is row equivalent to the $n \times n$ identity matrix.*

(c) *A has n pivot positions, i.e., A has pivot positions in each column (or row).*

(d) *The equation $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution.*

(e) *The columns of A form a linearly independent set.*

(f) *The linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ $(\boldsymbol{x} \mapsto A\boldsymbol{x})$ is one-to-one.*

(g) *The equation $A\boldsymbol{x} = \boldsymbol{b}$ has at least one solution for each $\boldsymbol{b} \in \mathbb{R}^n$.*

(h) *The columns of A span $\mathbb{R}^n$.*

(i) *The linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ $(\boldsymbol{x} \mapsto A\boldsymbol{x})$ is onto.*

(j) *There is an $n \times n$ matrix C such that $CA = I$.*

(k) *There is an $n \times n$ matrix D such that $AD = I$.*

(l) *$A^\top$ is an invertible matrix.*

Corollary 6.2 (page 130) *Let A and B be square matrices of size n.*

(a) *Suppose $AB = I$. Then $BA = I$. In particular, both A and B are invertible and $B = A^{-1}$, $A = B^{-1}$.*

(b) *AB is invertible if and only if both A and B are invertible.*

Note. If $AB = I_m$ for an $m \times n$ matrix A and an $n \times m$ matrix B. Then $m \leq n$. In particular, if $AB = I_m$ and $BA = I_n$, then $m = n$.

Theorem 6.3 (Thoerem 9 in page 132) *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let A be the standard matrix for T. Then there is a function $S : \mathbb{R}^n \to \mathbb{R}^n$ satisfying $S(T(\boldsymbol{x})) = \boldsymbol{x}$ and $T(S(\boldsymbol{x})) = \boldsymbol{x}$ for all $\boldsymbol{x} \in \mathbb{R}^n$ if and only if A is invertible. In this case A^{-1} is the standard matrix of S.*

6.3 Partitioned Matrices

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} W & X \\ Y & Z \end{bmatrix} = \begin{bmatrix} AW + BY & AX + BZ \\ CW + DY & CX + DZ \end{bmatrix}.$$

Theorem 6.4 (Theorem 10 (Column-Row Expansion of AB, page 137)) *If A is $m \times n$ and B is $n \times p$, then*

$$\begin{aligned} AB &= [\mathrm{col}_1(A), \mathrm{col}_2(A), \cdots, \mathrm{col}_n(A)] \begin{bmatrix} \mathrm{row}_1(B) \\ \mathrm{row}_2(B) \\ \vdots \\ \mathrm{row}_n(B) \end{bmatrix} \\ &= \mathrm{col}_1(A)\mathrm{row}_1(B) + \mathrm{col}_2(A)\mathrm{row}_2(B) + \cdots + \mathrm{col}_n(A)\mathrm{row}_n(B). \end{aligned}$$