3 Vectors

3.1 Vectors in $\mathbb{R}^n$

Definition 3.1 (page 20) If m and n are positive integers（正の整数）, an $m \times n$ *matrix*（行列） is a rectangular array（長方形に並んだ）of numbers with m rows（行）and n columns（列）. If n is a positive integer, an $n \times 1$ matrix is often called an n-*dimensional column vector* and a $1 \times n$ matrix an n-*dimensional row vector*. The collection of all n-dimensional column (or row) vectors is denoted by $\mathbb{R}^n$.

The vector whose entries（成分）are all zero is called the *zero vector* and is denoted by $\mathbf{0}$. The number of entries in $\mathbf{0}$ will be clear from the context.

Equality of (column or row) vectors in $\mathbb{R}^n$ and the operations of scalar multiplication（スカラー倍）and vector addition（ベクトルの和）in $\mathbb{R}^n$ are defined entry by entry. Thus for $c \in \mathbb{R}$ and

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \ \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n, \ c\boldsymbol{u} = c\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} cu_1 \\ cu_2 \\ \vdots \\ cu_n \end{bmatrix}, \ \boldsymbol{u}+\boldsymbol{v} = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ \vdots \\ u_n+v_n \end{bmatrix}.$$

Example 3.1 A is a 3×4 matrix, $\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}'', \boldsymbol{u}'''$ are (3-dimensional) column vectors in $\mathbb{R}^3$, $\boldsymbol{v}, \boldsymbol{v}', \boldsymbol{v}''$ are (4-dimensional) row vectors, and $\boldsymbol{w}$ is a (3-dimensional) row vector.

$$A = \begin{bmatrix} 3 & 1 & 2 & 4 \\ 1 & 1 & 1 & 1 \\ 11 & -1 & 5 & 17 \end{bmatrix}, \ \boldsymbol{u} = \begin{bmatrix} 3 \\ 1 \\ 11 \end{bmatrix}, \boldsymbol{u}' = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \boldsymbol{u}'' = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}, \boldsymbol{u}''' = \begin{bmatrix} 4 \\ 1 \\ 17 \end{bmatrix},$$

$$\boldsymbol{v} = [3,1,2,4], \boldsymbol{v}' = [1,1,1,1], \boldsymbol{v}'' = [11,-1,5,17], \boldsymbol{w} = [3,1,11].$$

In order to save space, a column vector such as $\boldsymbol{u}$ above is written in the following way as well.

$$\boldsymbol{u} = [3,1,11]^\top = \boldsymbol{w}^\top \text{ the } \textit{transpose} \text{ of } \boldsymbol{w}.$$

$$(\boldsymbol{u} + (-1)\boldsymbol{u}') + \boldsymbol{u}'' = \left(\begin{bmatrix} 3 \\ 1 \\ 11 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right) + \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 17 \end{bmatrix} = \boldsymbol{u}''',$$

$$\boldsymbol{v} + (-3)\boldsymbol{v}' = [3,1,2,4] + (-3)[1,1,1,1] = [3,1,2,4] + [-3,-3,-3,-3] = [0,-2,-1,1].$$

Algebraic Properties of $\mathbb{R}^n$ (page 43) For all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$ and all scalars c and d:

(i)	$\boldsymbol{u}+\boldsymbol{v} = \boldsymbol{v}+\boldsymbol{u}$	(v)	$c(\boldsymbol{u}+\boldsymbol{v}) = c\boldsymbol{u}+c\boldsymbol{v}$
(ii)	$(\boldsymbol{u}+\boldsymbol{v})+\boldsymbol{w} = \boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w})$	(vi)	$(c+d)\boldsymbol{u} = c\boldsymbol{u}+d\boldsymbol{u}$
(iii)	$\boldsymbol{u}+\mathbf{0} = \boldsymbol{u} = \mathbf{0}+\boldsymbol{u}$	(vii)	$c(d\boldsymbol{u}) = (cd)\boldsymbol{u}$
(iv)	$\boldsymbol{u}+(-\boldsymbol{u}) = \mathbf{0} = (-\boldsymbol{u})+\boldsymbol{u}$ where $-\boldsymbol{u}$ denotes $(-1)\boldsymbol{u}$	(viii)	$1\boldsymbol{u} = \boldsymbol{u}$

For simplicity of notation, a vector such as $\boldsymbol{u} + (-1)\boldsymbol{v}$ is often written as $\boldsymbol{u} - \boldsymbol{v}$. These properties are satisfied by row vectors as well.

Definition 3.2 For (column (or row)) vectors

$$\boldsymbol{u} = [u_1, u_2, \ldots, u_n]^\top,\ \boldsymbol{v} = [v_1, v_2, \ldots, v_n]^\top \in \mathbb{R}^n,$$

the *inner product*（内積）of $\boldsymbol{u}$ and $\boldsymbol{v}$ is defined by:

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

When $\boldsymbol{u} \cdot \boldsymbol{v} = 0$, $\boldsymbol{u}, \boldsymbol{v}$ are called *orthogonal* (or *perpendicular* 直交する). The *norm*（ノルム）of $\boldsymbol{u}$ is $\|\boldsymbol{u}\| = \sqrt{\boldsymbol{u} \cdot \boldsymbol{u}} = \sqrt{{u_1}^2 + {u_2}^2 + \cdots + {u_n}^2}$.

For $\boldsymbol{u} = [u_1, u_2, \ldots, u_n]^\top, \boldsymbol{v} = [v_1, v_2, \ldots, v_n]^\top, \boldsymbol{w} = [w_1, w_2, \ldots, w_n]^\top \in \mathbb{R}^n$ and $c \in \mathbb{R}$,

$$\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u},\ (\boldsymbol{u} + \boldsymbol{v}) \cdot \boldsymbol{w} = \boldsymbol{u} \cdot \boldsymbol{w} + \boldsymbol{v} \cdot \boldsymbol{w},\ (c\boldsymbol{u}) \cdot \boldsymbol{v} = c(\boldsymbol{u} \cdot \boldsymbol{v}) = \boldsymbol{u} \cdot (c\boldsymbol{v}).$$

Moreover, $\|\boldsymbol{u}\| \geq 0$ and $\|\boldsymbol{u}\| = 0$ if and only if $\boldsymbol{u} = \boldsymbol{0}$.

$$\begin{aligned}
(\boldsymbol{u} + \boldsymbol{v}) \cdot \boldsymbol{w} &= ([u_1, u_2, \ldots, u_n]^\top + [v_1, v_2, \ldots, v_n]^\top) \cdot [w_1, w_2, \ldots, w_n]^\top \\
&= [u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n]^\top \cdot [w_1, w_2, \ldots, w_n]^\top \\
&= (u_1 + v_1)w_1 + (u_2 + v_2)w_2 + \cdots + (u_n + v_n)w_n \\
&= u_1 w_1 + u_2 w_2 + \cdots + u_n w_n + v_1 w_1 + v_2 w_2 + \cdots + v_n w_n \\
&= \boldsymbol{u} \cdot \boldsymbol{w} + \boldsymbol{v} \cdot \boldsymbol{w}.
\end{aligned}$$

Theorem 3.1 (Cauchy-Schwarz Inequality in pages 377-8) *For* $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$

$$-\|\boldsymbol{u}\|\|\boldsymbol{v}\| \leq \boldsymbol{u} \cdot \boldsymbol{v} \leq \|\boldsymbol{u}\|\|\boldsymbol{v}\|$$

Equality holds if and only if one of $\boldsymbol{u}$ *and* $\boldsymbol{v}$ *is a scalar multiple of the other.*

Proof. We may assume that $\boldsymbol{u} \neq \boldsymbol{0}$ is a non-zero vector in $\mathbb{R}^n$. Then for any real number λ,

$$0 \leq \|\lambda\boldsymbol{u} + \boldsymbol{v}\|^2 = (\lambda\boldsymbol{u} + \boldsymbol{v}) \cdot (\lambda\boldsymbol{u} + \boldsymbol{v}) = \lambda^2\|\boldsymbol{u}\|^2 + 2(\boldsymbol{u} \cdot \boldsymbol{v})\lambda + \|\boldsymbol{v}\|^2.$$

It follows from a property of quadratic equations（2次方程式）, $(\boldsymbol{u} \cdot \boldsymbol{v})^2 - \|\boldsymbol{u}\|^2\|\boldsymbol{v}\|^2 \leq 0$. Since $\|\boldsymbol{u}\| \geq 0$ and $\|\boldsymbol{v}\| \geq 0$, $|\boldsymbol{u} \cdot \boldsymbol{v}| \leq \|\boldsymbol{u}\|\|\boldsymbol{v}\|$ and the inequalities hold.

Definition 3.3 Suppose $\boldsymbol{u} \neq \boldsymbol{0}$ and $\boldsymbol{v} \neq \boldsymbol{0}$. Then the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ are a real number θ such that $0 \leq \theta \leq \pi$ satisfying

$$\cos\theta = \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}.$$

Definition 3.4 Let $\boldsymbol{u} = [u_1, u_2, u_3]^\top, \boldsymbol{v} = [v_1, v_2, v_3]^\top$ be vectors in $\mathbb{R}^3$. Then the *vector product* (or *exterior product* ベクトル積、外積) of $\boldsymbol{u}$ and $\boldsymbol{v}$ is:

$$\begin{aligned}
\boldsymbol{u} \times \boldsymbol{v} &= [u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1]^\top \\
&= \left[\begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix}, \begin{vmatrix} u_3 & v_3 \\ u_1 & v_1 \end{vmatrix}, \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix}\right]^\top,\ \text{where} \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc.
\end{aligned}$$

The following hold: $\boldsymbol{u} \times \boldsymbol{v} = -\boldsymbol{v} \times \boldsymbol{u},\ \boldsymbol{u} \cdot (\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{v} \cdot (\boldsymbol{u} \times \boldsymbol{v}) = 0.$

3.2 Exercises

1. For (i) and (ii), compute each of (a) - (e) below.

   (a) $\boldsymbol{u} \cdot \boldsymbol{v}$ (b) $\boldsymbol{u} \cdot (\boldsymbol{v} + \boldsymbol{w})$ (c) $\|\boldsymbol{u}\|, \|\boldsymbol{v}\|, \|\boldsymbol{w}\|$

   (d) angle between $\boldsymbol{v}$ and $\boldsymbol{w}$ (e) a nonzero $\boldsymbol{x}$ such that $\boldsymbol{x} \cdot \boldsymbol{u} = \boldsymbol{x} \cdot \boldsymbol{v} = \boldsymbol{x} \cdot \boldsymbol{w} = 0$

   (i) $\boldsymbol{u} = [3, 1, 11]^\top, \boldsymbol{v} = [1, 1, -1]^\top$, and $\boldsymbol{w} = [2, 1, 5]^\top$.

   (ii) $\boldsymbol{u} = [1, 1, 1, 1]^\top, \boldsymbol{v} = [1, 1, -1, -1]^\top$, and $\boldsymbol{w} = [1, -1, 1, -1]^\top$.

2. For all vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$, show the following.

   (a) $\|\boldsymbol{u} + \boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2 + 2(\boldsymbol{u} \cdot \boldsymbol{v}) + \|\boldsymbol{v}\|^2$.

   (b) (The Triangular Inequlity) $\|\boldsymbol{u} + \boldsymbol{v}\| \leq \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$.

   (c) $\|\boldsymbol{u} + \boldsymbol{v}\| = \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$ if and only if one of the vectors is a positive scalar multiple of the other.

   (d) (Pythagoras' Theorem) $\|\boldsymbol{u} + \boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2$ if and only if $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal.

3. Let $\boldsymbol{u} = [3, 2, -1]^\top, \boldsymbol{v} = [0, 2, -3]^\top$, and $\boldsymbol{w} = [2, 6, 7]^\top$. Compute

   (a) $\boldsymbol{v} \times \boldsymbol{w}$ (b) $\boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{w})$ (c) $(\boldsymbol{u} \times \boldsymbol{v}) \times \boldsymbol{w}$

   (d) $(\boldsymbol{u} \times \boldsymbol{v}) \times (\boldsymbol{v} \times \boldsymbol{w})$ (e) $\boldsymbol{u} \times (\boldsymbol{v} - 2\boldsymbol{w})$ (f) $(\boldsymbol{u} \times \boldsymbol{v}) - 2\boldsymbol{w}$

4. Let $\boldsymbol{u} = [u_1, u_2, u_3]^\top, \boldsymbol{v} = [v_1, v_2, v_3]^\top$ be vectors in $\mathbb{R}^3$. Show

   (a) $\boldsymbol{u} \times \boldsymbol{v} = -\boldsymbol{v} \times \boldsymbol{u}$.

   (b) $\boldsymbol{u} \cdot (\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{v} \cdot (\boldsymbol{u} \times \boldsymbol{v}) = 0$.

5. Find a vector that is orthogonal to both $\boldsymbol{u}$ and $\boldsymbol{v}$.

   (a) $\boldsymbol{u} = [-6, 4, 2]^\top,\ \boldsymbol{v} = [3, 1, 5]^\top$ (b) $\boldsymbol{u} = [-2, 1, 5]^\top,\ \boldsymbol{v} = [3, 0, -3]^\top$

6. Find the scalar triple product $\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})$.

   (a) $\boldsymbol{u} = [-1, 2, 4]^\top,\ \boldsymbol{v} = [3, 4, -2]^\top,\ \boldsymbol{w} = [-1, 2, 5]^\top$.

   (b) $\boldsymbol{u} = [3, -1, 6]^\top,\ \boldsymbol{v} = [2, 4, 3]^\top,\ \boldsymbol{w} = [5, -1, 2]^\top$.

7. Suppose that $\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) = 3$. Find

   (a) $\boldsymbol{u} \cdot (\boldsymbol{w} \times \boldsymbol{v})$ (b) $(\boldsymbol{v} \times \boldsymbol{w}) \cdot \boldsymbol{u}$ (c) $\boldsymbol{w} \cdot (\boldsymbol{u} \times \boldsymbol{v})$

   (d) $\boldsymbol{v} \cdot (\boldsymbol{u} \times \boldsymbol{w})$ (e) $(\boldsymbol{u} \times \boldsymbol{w}) \cdot \boldsymbol{v}$ (f) $\boldsymbol{v} \cdot (\boldsymbol{w} \times \boldsymbol{w})$

8. Let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ be vectors in $\mathbb{R}^3$. Show the following.

   (a) $\|\boldsymbol{u} \times \boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2\|\boldsymbol{v}\|^2 - (\boldsymbol{u} \cdot \boldsymbol{v})^2$.

   (b) $\boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{w}) = (\boldsymbol{u} \cdot \boldsymbol{w})\boldsymbol{v} - (\boldsymbol{u} \cdot \boldsymbol{v})\boldsymbol{w}$, and $(\boldsymbol{u} \times \boldsymbol{v}) \times \boldsymbol{w} = (\boldsymbol{u} \cdot \boldsymbol{w})\boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{w})\boldsymbol{u}$.

First try Exercises 1 (i), (ii) and 3 (a), (b).

3.3 Related Problems in Final Exams

Final 2012 1(b) Find $\boldsymbol{v}_1 \times \boldsymbol{v}_2$, where $\boldsymbol{v}_1 = [3, 2, -3]^\top$ and $\boldsymbol{v}_2 = [1, 2, 1]^\top$.

[Soln. $\boldsymbol{v}_1 \times \boldsymbol{v}_2 = [8, -6, 4]^\top$]

Final 2013 1(c) Find $\boldsymbol{v}_1 \times \boldsymbol{v}_2$, where $\boldsymbol{v}_1 = [3, 1, -2]^\top$ and $\boldsymbol{v}_2 = [1, 2, 4]^\top$.

[Soln. $\boldsymbol{v}_1 \times \boldsymbol{v}_2 = [8, -14, 5]^\top$]

Final 2014 1 Let $\boldsymbol{u} = [2, 1, -3]^\top$, $\boldsymbol{v} = [0, 1, 2]^\top$, $\boldsymbol{w} = [1, 3, 1]^\top$.

(a) Find $\boldsymbol{u} \times \boldsymbol{v}$. [Soln. $[5, -4, 2]^\top$]

(b) Find $(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w}$. Note that $|(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w}|$ is the volume of a parallelepiped defined by $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$. [Soln. -5.]

Final 2015 1 Let $\boldsymbol{u} = [1, 4, 9]^\top$, $\boldsymbol{v} = [1, 8, 27]^\top$, $\boldsymbol{w} = [1, 2, 3]^\top$.

(a) Find $\boldsymbol{u} \times \boldsymbol{v}$. [Soln. $[36, -18, 4]^\top$]

(b) Find $(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w}$. Note that $|(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w}|$ is the volume of a parallelepiped defined by $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$. [Soln. 12.]

Final 2010 1(a) Let $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ be as follows.

$$\boldsymbol{u} = [4, -8, 1]^t op, \ \boldsymbol{v} = [2, 1, -2]^\top, \ \boldsymbol{w} = [3, -4, 12]^\top.$$

The vector $\boldsymbol{p} = \mathrm{proj}_{\boldsymbol{v}} \boldsymbol{u}$ is a scalar multiple of $\boldsymbol{v}$ such that $\boldsymbol{u} - \boldsymbol{p}$ is orthogonal to $\boldsymbol{v}$. Find $\boldsymbol{p}$. [Soln. $[-4/9, -2/9, 4/9]^\top$]

Proposition. Let $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ be vectors in $\mathbb{R}^3$. Then $\|\boldsymbol{u} \times \boldsymbol{v}\|$ is the area of the parallelogram（平行四辺形） determined by $\boldsymbol{u}$ and $\boldsymbol{v}$ and $|(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w}|$ is the volume of the parallelepiped（平行六面体）determined by $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$.