2 Solution Sets of Linear Equations

We look at a system of linear equations in two different ways, i.e., a vector equation and a matrix equation.

2.1 Vector Equations

Vectors: Recall that if m and n are positive integers（正の整数）, an $m \times n$ *matrix*（行列） is a rectangular array（長方形に並んだ）of numbers with m rows（行）and n columns（列）. An $n \times 1$ matrix is often called an *n-dimensional column vector*（n-次元列ベクトル）, and a $1 \times n$ matrix an *n-dimensional row vector*（n-次元行ベクトル）. The collection of all n-dimensional column (or row) vectors is denoted by $\mathbb{R}^n$.

Scalar Multiple and Sum of Vectors: For $c \in \mathbb{R}$（c を実数とし） and

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix},\ \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n,\ c\boldsymbol{u} = c\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} cu_1 \\ cu_2 \\ \vdots \\ cu_n \end{bmatrix},\ \boldsymbol{u}+\boldsymbol{v} = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ \vdots \\ u_n+v_n \end{bmatrix}.$$

Example 2.1

$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ -2 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix} + (-2)\begin{bmatrix} 1 \\ 0 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 3 \\ -4 \end{bmatrix}$$

Example 2.2

$$\left\{\begin{array}{rcl} x_1 + 0x_2 + x_3 + 0x_4 + x_5 + 3x_6 & = & -1 \\ -x_1 + 0x_2 - x_3 + 0x_4 + 0x_5 - 4x_6 & = & -1 \\ 0x_1 + x_2 - 2x_3 + 3x_4 + 0x_5 - x_6 & = & 3 \\ -2x_1 - 2x_2 + 2x_3 - 6x_4 - 2x_5 - 4x_6 & = & -4 \end{array}\right. \quad \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 3 & -1 \\ -1 & 0 & -1 & 0 & 0 & -4 & -1 \\ 0 & 1 & -2 & 3 & 0 & -1 & 3 \\ -2 & -2 & 2 & -6 & -2 & -4 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ -2 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix} + 0\begin{bmatrix} 1 \\ -1 \\ -2 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 0 \\ 3 \\ -6 \end{bmatrix} + (-2)\begin{bmatrix} 1 \\ 0 \\ 0 \\ -2 \end{bmatrix} + 0\begin{bmatrix} 3 \\ -4 \\ -1 \\ -4 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 3 \\ -4 \end{bmatrix} = \boldsymbol{b}$$

$$x_1\begin{bmatrix} 1 \\ -1 \\ 0 \\ -2 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix} + x_3\begin{bmatrix} 1 \\ -1 \\ -2 \\ 2 \end{bmatrix} + x_4\begin{bmatrix} 0 \\ 0 \\ 3 \\ -6 \end{bmatrix} + x_5\begin{bmatrix} 1 \\ 0 \\ 0 \\ -2 \end{bmatrix} + x_6\begin{bmatrix} 3 \\ -4 \\ -1 \\ -4 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + 0x_2 + x_3 + 0x_4 + x_5 + 3x_6 \\ -x_1 + 0x_2 - x_3 + 0x_4 + 0x_5 - 4x_6 \\ 0x_1 + x_2 - 2x_3 + 3x_4 + 0x_5 - x_6 \\ -2x_1 - 2x_2 + 2x_3 - 6x_4 - 2x_5 - 4x_6 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 3 \\ -4 \end{bmatrix}.$$

Let

$$\boldsymbol{a}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ -2 \end{bmatrix},\ \boldsymbol{a}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix},\ \boldsymbol{a}_3 = \begin{bmatrix} 1 \\ -1 \\ -2 \\ 2 \end{bmatrix},\ \boldsymbol{a}_4 = \begin{bmatrix} 0 \\ 0 \\ 3 \\ -6 \end{bmatrix},\ \boldsymbol{a}_5 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -2 \end{bmatrix},\ \boldsymbol{a}_6 = \begin{bmatrix} 3 \\ -4 \\ -1 \\ -4 \end{bmatrix}.$$

Then

$$\boldsymbol{a}_1 + 3\boldsymbol{a}_2 + 0\boldsymbol{a}_3 + 0\boldsymbol{a}_4 + (-2)\boldsymbol{a}_5 + 0\boldsymbol{a}_6 = \boldsymbol{b}.$$

Hence $(x_1, x_2, x_3, x_4, x_5, x_6) = (1, 3, 0, 0, -2, 0)$ is a solution to a vector equation

$$x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + x_3\boldsymbol{a}_3 + x_4\boldsymbol{a}_4 + x_5\boldsymbol{a}_5 + x_6\boldsymbol{a}_6 = \boldsymbol{b}.$$

Please check algebraic properties of $\mathbb{R}^n$ in page 43.

Algebraic Properties of $\mathbb{R}^n$ [page 43] For all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$ and all scalars c and d:

(i) $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$.

(ii) $(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w})$.

(iii) $\boldsymbol{u} + \boldsymbol{0} = \boldsymbol{0} + \boldsymbol{u} = \boldsymbol{u}$.

(iv) $\boldsymbol{u} + (-\boldsymbol{u}) = (-\boldsymbol{u}) + \boldsymbol{u} = \boldsymbol{0}$, where $-\boldsymbol{u}$ denotes $(-1)\boldsymbol{u}$.

(v) $c(\boldsymbol{u} + \boldsymbol{v}) = c\boldsymbol{u} + c\boldsymbol{v}$.

(vi) $(c + d)\boldsymbol{u} = c\boldsymbol{u} + d\boldsymbol{v}$.

(vii) $c(d\boldsymbol{u}) = (cd)\boldsymbol{v}$.

(viii) $1\boldsymbol{u} = \boldsymbol{u}$.

Definition 2.1 [page 44] Given vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ in $\mathbb{R}^m$, and given scalars $c_1, c_2, \ldots, c_n$, the vector $\boldsymbol{y}$ defined by

$$\boldsymbol{y} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_n\boldsymbol{v}_n$$

is called a *linear combination* (一次 (線形) 結合) of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ with *weights* $c_1, c_2, \ldots, c_n$.

Consider

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{cases}$$

Let

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix},\ \boldsymbol{a}_1 = \begin{bmatrix} a_{1,1} \\ a_{2,1} \\ \vdots \\ a_{m,1} \end{bmatrix}, \ldots, \boldsymbol{a}_n = \begin{bmatrix} a_{1,n} \\ a_{2,n} \\ \vdots \\ a_{m,n} \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Then

$$x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_n\boldsymbol{a}_n$$
$$= x_1 \begin{bmatrix} a_{1,1} \\ a_{2,1} \\ \vdots \\ a_{m,1} \end{bmatrix} + x_2 \begin{bmatrix} a_{1,2} \\ a_{2,2} \\ \vdots \\ a_{m,2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1,n} \\ a_{2,n} \\ \vdots \\ a_{m,n} \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

Definition 2.2 [page 46] If $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_p$ are in $\mathbb{R}^n$, the set of all linear combinations of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_p$ is denoted by $\text{Span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_p\}$. That is, $\text{Span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_p\}$ is the collection of all vectors that can be written in the form

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_p\boldsymbol{v}_p$$

with $c_1, c_2, \ldots, c_p$ scalars.

Proposition 2.1 (page 46) *A vector equation*

$$x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_n\boldsymbol{a}_n = \boldsymbol{b}$$

has the same solution set as the linear system whose augmented matrix is

$$[\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n, \boldsymbol{b}]$$

The system is consistent if and only if the vector $\boldsymbol{b}$ *is a linear combination of vectors* $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$*, if and only if* $\boldsymbol{b}$ *is in* $\text{Span}\{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n\}$.

2.2 Matrix Equation $A\boldsymbol{x} = \boldsymbol{b}$

Definition 2.3 [page 51] If A is an $m \times n$ matrix with columns $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$, and if $\boldsymbol{x} \in \mathbb{R}^n$, then the product of A and $\boldsymbol{x}$, denoted by $A\boldsymbol{x}$ is *the linear combination of the columns of A using the corresponding entries in $\boldsymbol{x}$ as weights*, that is

$$A\boldsymbol{x} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_n\boldsymbol{a}_n.$$

Proposition 2.2 (Theorem 5 in page 55) *If A is an $m \times n$ matrix, $\boldsymbol{u}, \boldsymbol{v}$ are vectors in $\mathbb{R}^n$, and c is a scalar, then:*

(a) $A(\boldsymbol{u} + \boldsymbol{v}) = A\boldsymbol{v} + A\boldsymbol{v}$.

(b) $A(c\boldsymbol{u}) = cA\boldsymbol{u}$.

Row-Vector Rule for Computing $A\boldsymbol{x}$ (page 54) If the product $A\boldsymbol{x}$ is defined, then the ith entry in $A\boldsymbol{x}$ is the sume of the products of corresponding entries from row i of A and from the vector $\boldsymbol{x}$.

Proposition 2.3 (Theorem 3 in page 52) *If A is an $m \times n$ matrix, with columns $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$, and if $\boldsymbol{v} \in \mathbb{R}^m$, the matrix equation*

$$A\boldsymbol{x} = \boldsymbol{b}$$

has the same solution set as the vector equation

$$x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_n\boldsymbol{a}_n = \boldsymbol{b}$$

which in tern, has the same solution set as the system of linear equations whose augmented matrix is

$$[\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n, \boldsymbol{b}].$$

Theorem 2.4 (Theorem 4 in page 53) *Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or they are all false.*

(a) *For each $\boldsymbol{b}$ in $\mathbb{R}^m$, the equation $A\boldsymbol{x} = \boldsymbol{b}$ has a solution.*

(b) *Each $\boldsymbol{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.*

(c) *The columns of A span $\mathbb{R}^m$.*

(d) *A has a pivot position in every row.*

2.3 Solution Sets of Linear System

Homogeneous Linear System (page 59) A linear system is said to be *homogeneous*（斉次） if it can be written in the form $A\boldsymbol{x} = \boldsymbol{0}$. Such a system always has at least one solution, namely $\boldsymbol{x} = \boldsymbol{0}$. The zero solution is usually called the *trivial solution*（自明な解）, and a nonzero solution is called a *nontrivial solution*（非自明な解）.

Proposition 2.5 (page 60) *The homogeneous equation $A\boldsymbol{x} = \boldsymbol{0}$ has a nontrivial solution if and only if the equation has at least one free variable.*

Theorem 2.6 *A homogeneous system of linear equations with more unknowns than equations has infinitely many solutions. In particular, if A is an $m \times n$ matrix with $m < n$, then a matrix equation $A\boldsymbol{x} = \boldsymbol{0}$ has a nonzero solution.*

Theorem 2.7 (Theorem 6 in page 63) *Suppose the equation $A\boldsymbol{x} = \boldsymbol{b}$ is consistent for some given $\boldsymbol{b}$, and let $\boldsymbol{p}$ be a solution. Then the solution set of $A\boldsymbol{x} = \boldsymbol{b}$ is the set of all vectors of the form $\boldsymbol{w} = \boldsymbol{p} + \boldsymbol{v}_h$, where $\boldsymbol{v}_h$ is any solution of the homogeneous equation $A\boldsymbol{x} = \boldsymbol{0}$.*