Take-Home Quiz 1 (*Due at 7:00 p.m. on Fri. September 14, 2007*)

Division: **ID#:** **Name:**

Let us consider the following system of linear equations in 6 unknowns $x_1, x_2, \ldots, x_6$.

$$\left\{\begin{array}{rcl} x_1 + x_3 - x_4 + 4x_5 & = & -3 \\ 2x_1 + 2x_3 - x_4 + 6x_5 & = & 1 \\ x_1 + x_3 + 2x_5 - x_6 & = & 5 \\ -x_1 - 2x_2 - 7x_3 - 4x_5 + x_6 & = & -7 \end{array}\right. \qquad B = \begin{bmatrix} 1 & 0 & 1 & -1 & 4 & 0 & -3 \\ 2 & 0 & 2 & -1 & 6 & 0 & 1 \\ 0 & -2 & -6 & 0 & -2 & 0 & -2 \\ -1 & -2 & -7 & 0 & -4 & 1 & -7 \end{bmatrix}$$

1. Find the augmented matrix A of the system of linear equations above.

2. The matrix B is obtained by applying an elementary row operation once to the augmented matrix A. Write the elementary row operation using the notation $[i; c]$, $[i, j]$, or $[i, j; c]$.

3. Find the reduced row echelon form of the augmented matrix A. (Solution only.)

4. Find the solution of the system of linear equations. Use parameters if necessary.

Message: (1) この授業に期待すること (2) あなたにとって数学とは [HP 掲載不可のときは明記のこと]

Solutions to Take-Home Quiz 1 (*September 14, 2007*)

$$\begin{cases} x_1 + x_3 - x_4 + 4x_5 & = -3 \\ 2x_1 + 2x_3 - x_4 + 6x_5 & = 1 \\ x_1 + x_3 + 2x_5 - x_6 & = 5 \\ -x_1 - 2x_2 - 7x_3 - 4x_5 + x_6 & = -7 \end{cases} \qquad B = \begin{bmatrix} 1 & 0 & 1 & -1 & 4 & 0 & -3 \\ 2 & 0 & 2 & -1 & 6 & 0 & 1 \\ 0 & -2 & -6 & 0 & -2 & 0 & -2 \\ -1 & -2 & -7 & 0 & -4 & 1 & -7 \end{bmatrix}$$

1. Find the augmented matrix A of the system of linear equations above.

   Sol.

$$A = \begin{bmatrix} 1 & 0 & 1 & -1 & 4 & 0 & -3 \\ 2 & 0 & 2 & -1 & 6 & 0 & 1 \\ 1 & 0 & 1 & 0 & 2 & -1 & 5 \\ -1 & -2 & -7 & 0 & -4 & 1 & -7 \end{bmatrix}$$

2. The matrix B is obtained by applying an elementary row operation once to the augmented matrix A. Write the elementary row operation using the notation $[i; c]$, $[i, j]$, or $[i, j; c]$.

   Sol. $[3, 4; 1]$.

3. Find the reduced row echelon form of the augmented matrix A. (Solution only.)

   Sol. Apply the following consecutively in this order:

$$[2, 1; -2],\ [4, 1; 1],\ [2, 3],\ [2, -\frac{1}{2}],\ [4, 2; 2],\ [4, 3; 1],\ [1, 3; 1].$$

   Then we have

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 2 & 0 & 4 \\ 0 & 1 & 3 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}.$$

   - There are many ways to obtain the reduced echelon form but the final matrix should be the same. When can we change the order of operations and when cannot?
   - Starting from the reduced row echelon form above, is it possible to obtain the matrix A back again by applying elementary row operations? Can you find the sequence of such elementary row operations from the one we obtained the reduced echelon form from A with a slight modification?

4. Find the solution of the system of linear equations. Use parameters if necessary.

   Sol.

$$\begin{cases} x_1 & = 4 - s - 2t \\ x_2 & = 1 - 3s - t \\ x_3 & = s \\ x_4 & = 7 + 2t \\ x_5 & = t \\ x_6 & = -1 \end{cases}, \text{ or } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 7 \\ 0 \\ -1 \end{bmatrix} + s \begin{bmatrix} -1 \\ -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ -1 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}.$$

   s and t are parameters.

Take-Home **Quiz 2** (*Due at 7:00 p.m. on Fri. September 21, 2007*)

Division: **ID#:** **Name:**

Let $A = [a_{h,i}]$ be an $r \times s$ matrix, $B = [b_{j,k}]$ an $s \times t$ matrix, $C = [c_{l,m}]$ a $t \times u$ matrix and let L and T be matrices given below.

$$L = \begin{bmatrix} 0 & 1 & 0 \\ 6 & 1 & 3 \\ 0 & 4 & 3 \end{bmatrix}, \quad \text{and} \quad T = \begin{bmatrix} 1 & 1 & 1 \\ 6 & -3 & 1 \\ 8 & 2 & -2 \end{bmatrix}.$$

1. What is the size of the matrix $(AB)C$.

2. Write the (h,k)-entry of AB.

   $(AB)_{h,k} =$

3. Write the (h,m)-entry of $(AB)C$.

   $((AB)C)_{h,m} =$

4. Compute the product LT. (Show work!)

5. Find a 3×3 matrix D such that $LT = TD$. (Solution only.)

Message 欄：(理系以外の人も含め）高校・大学における数学は何のため？ [HP 掲載不可は明記のこと]

Solutions to Take-Home Quiz 2 (*September 21, 2007*)

1. What is the size of the matrix $(AB)C$.

   Sol. The matrix AB is of size $r \times t$ and C is of size $t \times u$. Hence the matrix $(AB)C$ is of size
$$r \times u.$$

2. Write the (h, k)-entry of AB.

   Sol.
$$\begin{aligned}(AB)_{h,k} &= A_{h,1}B_{1,k} + A_{h,2}B_{2,k} + \cdots + A_{h,s}B_{s,k} \\ &= a_{h,1}b_{1,k} + a_{h,2}b_{2,k} + \cdots + a_{h,s}b_{s,k} \\ &= \sum_{i=1}^{s} a_{h,i}b_{i,k} = \sum_{j=1}^{s} a_{h,j}b_{j,k}.\end{aligned}$$

3. Write the (h, m)-entry of $(AB)C$.

   Sol.
$$\begin{aligned}((AB)C)_{h,m} &= (AB)_{h,1}C_{1,m} + (AB)_{h,2}C_{2,m} + \cdots + (AB)_{h,t}C_{t,m} \\ &= \left(\sum_{i=1}^{s} a_{h,i}b_{i,1}\right) c_{1,m} + \left(\sum_{i=1}^{s} a_{h,i}b_{i,2}\right) c_{2,m} + \cdots + \left(\sum_{i=1}^{s} a_{h,i}b_{i,t}\right) c_{t,m} \\ &= \sum_{k=1}^{t}\left(\sum_{i=1}^{s} a_{h,i}b_{i,k}\right) c_{k,m} = \sum_{k=1}^{t}\sum_{i=1}^{s} a_{h,i}b_{i,k}c_{k,m}.\end{aligned}$$

   Note: From the above, we can show that $(AB)C = A(BC)$.

4. Compute the product LT. (Show work!)

   Sol.
$$\begin{aligned}LT &= \begin{bmatrix} 0 & 1 & 0 \\ 6 & 1 & 3 \\ 0 & 4 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 6 & -3 & 1 \\ 8 & 2 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 0\cdot1+1\cdot6+0\cdot8 & 0\cdot1+1\cdot(-3)+0\cdot2 & 0\cdot1+1\cdot1+0\cdot(-2) \\ 6\cdot1+1\cdot6+3\cdot8 & 6\cdot1+1\cdot(-3)+3\cdot2 & 6\cdot1+1\cdot1+3\cdot(-2) \\ 0\cdot1+4\cdot6+3\cdot8 & 0\cdot1+4\cdot(-3)+3\cdot2 & 0\cdot1+4\cdot1+3\cdot(-2) \end{bmatrix} \\ &= \begin{bmatrix} 6 & -3 & 1 \\ 36 & 9 & 1 \\ 48 & -6 & -2 \end{bmatrix}\end{aligned}$$

5. Find a 3×3 matrix D such that $LT = TD$. (Solution only.)

   Sol.
$$D = \begin{bmatrix} 6 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

   Note: Can you guess $AAT = A^2T$, $A^{10}T$ and A^nT?

Take-Home **Quiz 3** (*Due at 7:00 p.m. on Fri. September 28, 2007*)

Division: **ID#:** **Name:**

Let A and B be 3×3 matrices given below, and $C = [\,A \mid I\,]$, where I is the identity matrix of size three.

$$A = \begin{bmatrix} -3 & 1 & -1 \\ -3 & 1 & -2 \\ -1 & 0 & -2 \end{bmatrix},\ B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \\ -3 & 1 & -1 \end{bmatrix},\ \text{ and } C = \begin{bmatrix} -3 & 1 & -1 & 1 & 0 & 0 \\ -3 & 1 & -2 & 0 & 1 & 0 \\ -1 & 0 & -2 & 0 & 0 & 1 \end{bmatrix}$$

We applied elementary row operations $[1,3]$, $[1;-1]$, $[2,1;3]$ to the matrix C in this order and obtained a matrix $[\,B \mid P\,]$, where B is a 3×3 matrix above and P is a 3×3 matrix.

1. Find the matrix P.

2. Find the reduced row echelon form of the matrix C. (Solution only.)

3. Find the inverse matrix of A. (Solution only.)

4. Express P^{-1} as a product of elementary matrices using the notation $P(i;c)$, $P(i,j)$ and $P(i,j;c)$.

Message 欄：将来の夢、目標、25 年後の自分について、世界について。[HP 掲載不可は明記のこと]

Solutions to Take-Home Quiz 3 (*September 28, 2007*)

Let A and B be 3×3 matrices given below, and $C = [\, A \mid I \,]$, where I is the identity matrix of size three.

$$A = \begin{bmatrix} -3 & 1 & -1 \\ -3 & 1 & -2 \\ -1 & 0 & -2 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \\ -3 & 1 & -1 \end{bmatrix}, \ \text{and } C = \begin{bmatrix} -3 & 1 & -1 & 1 & 0 & 0 \\ -3 & 1 & -2 & 0 & 1 & 0 \\ -1 & 0 & -2 & 0 & 0 & 1 \end{bmatrix}$$

We applied elementary row operations $[1,3]$, $[1;-1]$, $[2,1;3]$ to the matrix C in this order and obtained a matrix $[\, B \mid P \,]$, where B is a 3×3 matrix above and P is a 3×3 matrix.

1. Find the matrix P.

   Sol.

$$I \xrightarrow{[1,3]} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \xrightarrow{[1;-1]} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \xrightarrow{[2,1;3]} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & -3 \\ 1 & 0 & 0 \end{bmatrix} = P$$

2. Find the reduced row echelon form of the matrix C. (Solution only.)

   Sol.

$$[B \mid P] = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & -1 \\ 0 & 1 & 4 & 0 & 1 & -3 \\ -3 & 1 & -1 & 1 & 0 & 0 \end{bmatrix} \xrightarrow{[3,1;3]} \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & -1 \\ 0 & 1 & 4 & 0 & 1 & -3 \\ 0 & 1 & 5 & 1 & 0 & -3 \end{bmatrix} \xrightarrow{[3,2;-1]}$$

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 0 & -1 \\ 0 & 1 & 4 & 0 & 1 & -3 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{bmatrix} \xrightarrow{[1,3;-2]} \begin{bmatrix} 1 & 0 & 0 & -2 & 2 & -1 \\ 0 & 1 & 4 & 0 & 1 & -3 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{bmatrix} \xrightarrow{[2,3;-4]}$$

$$\begin{bmatrix} 1 & 0 & 0 & -2 & 2 & -1 \\ 0 & 1 & 0 & -4 & 5 & -3 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{bmatrix} = [I \mid A^{-1}] \quad \text{(Reduced Echelon Form)}$$

3. Find the inverse matrix of A. (Solution only.)

   Sol.

$$A^{-1} = \begin{bmatrix} -2 & 2 & -1 \\ -4 & 5 & -3 \\ 1 & -1 & 0 \end{bmatrix}.$$

4. Express P^{-1} as a product of elementary matrices using the notation $P(i;c)$, $P(i,j)$ and $P(i,j;c)$.

   Sol. Since $P = P(2,1;3)P(1;-1)P(1,3)$,

$$P^{-1} = P(1,3)^{-1}P(1;-1)^{-1}P(2,1;3)^{-1} = P(1,3)P(1;-1)P(2,1;-3).$$

Take-Home **Quiz 4** (*Due at 7:00 p.m. on Fri. October 5, 2007*)

Division: **ID#:** **Name:**

(This quiz is designed to give you hints to read an article titled "The Reduced Row Echelon Form of a Matrix Is Unique: A Simple Proof," handed out at the second lecture.)

1. Express, if possible, the matrix below as a product of elementary matrices, if not, explain the reason. (If you apply a theorem, clarify which part is used.)

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 7 \\ 3 & 3 & 9 \end{bmatrix}$$

2. We want to show "the reduced row echelon form of a matrix is unique." Let A be an $m \times n$ matrix and let both B and C be reduced row echelon form of A. Since B and C are obtained by performing a series to elementary row operations to A, there are invertible matrices P and Q such that $B = PA$ and $C = QA$.

   (a) Let $\boldsymbol{x}$ be an $n \times 1$ matrix. Show that $A\boldsymbol{x} = \mathbf{0} \Leftrightarrow B\boldsymbol{x} = \mathbf{0}$, where $\mathbf{0}$ is the zero matrix of size $n \times 1$.

   (b) Let $\boldsymbol{x}$ be an $n \times 1$ matrix. Show that if $A\boldsymbol{x} = \mathbf{0}$, then $(B - C)\boldsymbol{x} = \mathbf{0}$.

Message 欄： あなたにとって、豊かな生活とはどのようなものでしょうか。どのようなとき幸せだと感じますか。[HP 掲載不可は明記のこと]

Solutions to Take-Home Quiz 4 *(October 5, 2007)*

(This quiz is designed to give you hints to read an article titled "The Reduced Row Echelon Form of a Matrix Is Unique: A Simple Proof," handed out at the second lecture.)

1. Express, if possible, the matrix below as a product of elementary matrices, if not, explain the reason. (If you apply a theorem, clarify which part is used.)

   Sol.

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 7 \\ 3 & 3 & 9 \end{bmatrix} \xrightarrow{[2,1;-2]} \begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & -1 \\ 3 & 3 & 9 \end{bmatrix} \xrightarrow{[3,1;-3]} \begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & -1 \\ 0 & -3 & -3 \end{bmatrix} \xrightarrow{[2;-1]} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & -3 & -3 \end{bmatrix}$$

$$\xrightarrow{[1,2;-2]} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & -3 & -3 \end{bmatrix} \xrightarrow{[3,2;3]} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{(Reduced row echelon form)}$$

   Now we apply Theorem 5.1 (1.5.3 or 1.6.4 in the textbook). (c) $\Leftrightarrow$ (d). Since the reduced row echelon form is not the identity matrix, A is not expressible as a product of elementary matrices. (Strictly speaking (d) $\Rightarrow$ (c), or its contraposition ((d) $\Rightarrow$ (c) の対偶) is used.)

   Another solution: Since $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 7 \\ 3 & 3 & 9 \end{bmatrix}\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, by Theorem 5.1 (1.5.3 or 1.6.4 in the textbook). (b) $\Leftrightarrow$ (d), the matrix cannot be expressed as a product of elementary matrices. (Strictly speaking (d) $\Rightarrow$ (b), or its contraposition is used. Can you tell how the nontrivial solution $[-2,-1,1]^T$ is found? It is read from the reduced row echelon form obtained above.) ■

2. We want to show "the reduced row echelon form of a matrix is unique." Let A be an $m \times n$ matrix and let both B and C be reduced row echelon form of A. Since B and C are obtained by performing a series to elementary row operations to A, there are invertible matrices P and Q such that $B = PA$ and $C = QA$.

   (a) Let $\boldsymbol{x}$ be an $n \times 1$ matrix. Show that $A\boldsymbol{x} = \mathbf{0} \Leftrightarrow B\boldsymbol{x} = \mathbf{0}$, where $\mathbf{0}$ is the zero matrix of size $n \times 1$.

   Sol. Suppose $A\boldsymbol{x} = \mathbf{0}$. Then $B\boldsymbol{x} = PA\boldsymbol{x} = P\mathbf{0} = \mathbf{0}$. Conversely suppose $B\boldsymbol{x} = \mathbf{0}$. Since P is invertible, $A\boldsymbol{x} = P^{-1}PA\boldsymbol{x} = P^{-1}B\boldsymbol{x} = P^{-1}\mathbf{0} = \mathbf{0}$. ■

   (b) Let $\boldsymbol{x}$ be an $n \times 1$ matrix. Show that if $A\boldsymbol{x} = \mathbf{0}$, then $(B - C)\boldsymbol{x} = \mathbf{0}$.

   Sol. By (a) we have $B\boldsymbol{x} = \mathbf{0} \Leftrightarrow A\boldsymbol{x} = \mathbf{0} \Leftrightarrow C\boldsymbol{x} = \mathbf{0}$, as $C = QA$ and Q is invertible. Suppose $A\boldsymbol{x} = \mathbf{0}$. Then $B\boldsymbol{x} = \mathbf{0} = C\boldsymbol{x}$. Hence $(B - C)\boldsymbol{x} = B\boldsymbol{x} - C\boldsymbol{x} = \mathbf{0} - \mathbf{0} = \mathbf{0}$. ■

Please read the article and understand its proof. It may be a little difficult but you can understand.

Take-Home **Quiz 5** (*Due at 7:00 p.m. on Fri. October 12, 2007*)

Division: **ID#:** **Name:**

Let A be the 4×4 matrix given below and B the submatrix that remains after 1st row and 2nd colum are deleted from A.

$$A = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 1 & 0 & -1 & -3 \\ 0 & -2 & 1 & 1 \\ 1 & 0 & -1 & -1 \end{bmatrix}, \ B = \begin{bmatrix} 1 & -1 & -3 \\ 0 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}.$$

Let $M_{i,j}$ be the minor of the (i, j) entry of A above, i.e., the determinant of the submatrix after ith row and jth column are deleted from A. In particular, $M_{1,2} = \det(B)$.

1. Find $\mathrm{adj}(B)$, the adjoint of B. (*Not* $\mathrm{adj}(A)$*!*)

2. Find $\det(B)$ and determine whether or not the matrix B is invertible.

3. Express $\det(A)$ by the cofactor expansion along the 1st row using minors $M_{i,j}$.

4. Express $\det(A)$ by the cofactor expansion along the 2nd column using minors $M_{i,j}$.

5. Find $\det(A)$.

Message 欄：これまでの Linear Algebra I について。改善点について。[HP 掲載不可は明記のこと]

Solutions to Take-Home Quiz 5 (*October 12, 2007*)

Let A be the 4×4 matrix given below and B the submatrix that remains after 1st row and 2nd colum are deleted from A.

$$A = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 1 & 0 & -1 & -3 \\ 0 & -2 & 1 & 1 \\ 1 & 0 & -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 & -3 \\ 0 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}.$$

Let $M_{i,j}$ be the minor of the (i,j) entry of A above, i.e., the determinant of the submatrix after ith row and jth column are deleted from A. In particular, $M_{1,2} = \det(B)$.

1. Find $\operatorname{adj}(B)$, the adjoint of B. (*Not* $\operatorname{adj}(A)$*!*)

   Sol. Let $m_{i,j}$ the minor of the (i,j) entry of B. Then

$$m_{1,1} = \det\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = 1(-1) - (-1)1 = 0, \; m_{1,2} = \det\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} = -1.$$

   Similarly, $m_{1,3} = -1$, $m_{2,1} = -2$, $m_{2,2} = 2$, $m_{2,3} = 0$, $m_{3,1} = 2$, $m_{3,2} = 1$, $m_{3,3} = 1$. Hence

$$\operatorname{adj}(B) = \begin{bmatrix} m_{1,1} & -m_{1,2} & m_{1,3} \\ -m_{2,1} & m_{2,2} & -m_{2,3} \\ m_{3,1} & -m_{3,2} & m_{3,3} \end{bmatrix}^T = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 2 & 0 \\ 2 & -1 & 1 \end{bmatrix}^T = \begin{bmatrix} 0 & 2 & 2 \\ 1 & 2 & -1 \\ -1 & 0 & 1 \end{bmatrix}.$$

2. Find $\det(B)$ and determine whether or not the matrix B is invertible.

   Sol.

$$\det(B) \cdot I = B \cdot \operatorname{adj}(B) = \begin{bmatrix} 1 & -1 & -3 \\ 0 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 2 \\ 1 & 2 & -1 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

   Hence $\det(B) = 2 \neq 0$. So B is invertible. Actually $B^{-1} = \frac{1}{2}\operatorname{adj}(B)$.

3. Express $\det(A)$ by the cofactor expansion along the 1st row using minors $M_{i,j}$.

   Sol. Let $C_{i,j}$ be the cofactor of the (i,j) entry of A. Then $C_{i,j} = (-1)^{i+j} M_{i,j}$. Hence

$$\det(A) = a_{1,1}C_{1,1} + a_{1,2}C_{1,2} + a_{1,3}C_{1,3} = 3M_{1,1} - 2M_{1,2} + M_{1,3}.$$

4. Express $\det(A)$ by the cofactor expansion along the 2nd column using minors $M_{i,j}$.

   Sol.

$$\det(A) = a_{1,2}C_{1,2} + a_{2,2}C_{2,2} + a_{3,2}C_{3,2} + a_{4,2}C_{4,2} = -2M_{1,2} + 2M_{3,2}.$$

5. Find $\det(A)$.

   Sol. Use the formula in 4. Since $M_{1,2} = 2$, it suffices to find $M_{3,2}$.

$$\det\begin{pmatrix} 3 & 1 & 0 \\ 1 & -1 & -3 \\ 1 & -1 & -1 \end{pmatrix} = 3((-1)(-1) - (-1)(-3)) - (1(-1) - (1)(-3)) = -8.$$

   Hence $\det(A) = 2M_{1,2} + 2M_{3,2} = -2 \cdot 2 + 2 \cdot (-8) = -20$.

Take-Home **Quiz 6** (*Due at 7:00 p.m. on Fri. October 19, 2007*)

Division: **ID#:** **Name:**

Let A, $\boldsymbol{x}$, $\boldsymbol{b}$ and T be as follows, where a, b, c and d are arbitrary numbers.

$$A = \begin{bmatrix} 2 & -2 & -4 & 0 \\ -3 & 5 & 4 & 5 \\ 4 & 2 & -5 & 3 \\ 5 & -7 & -3 & 0 \end{bmatrix}, \ \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \ \boldsymbol{b} = \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \text{ and } T = \begin{bmatrix} a & b & c & c \\ b & a & c & c \\ c & c & a & b \\ c & c & b & a \end{bmatrix}$$

1. In the following we consider the equation $A\boldsymbol{x} = \boldsymbol{b}$.

   (a) Evaluate $\det(A)$, and determine whether there is no solution, exactly one solution or infinitely many solutions.

   (b) By Cramer's rule express $x_3 = \frac{\det(B)}{\det(A)}$ as a fraction of two determinants. Write down the matrix B in the numerator.

   (c) Evaluate $\det(B)$ in the previous problem and find x_3.

2. Evaluate the determinant of T.

Message 欄：数学（または他の科目）など何かを学んでいて感激したことについて。
[HP 掲載不可は明記のこと]

Solutions to Take-Home Quiz 6 (*October 19, 2007*)

1. In the following we consider the equation $A\boldsymbol{x} = \boldsymbol{b}$.

   (a) Evaluate $\det(A)$, and determine whether there is no solution, exactly one solution or infinitely many solutions.

   Sol.

$$\begin{vmatrix} 2 & -2 & -4 & 0 \\ -3 & 5 & 4 & 5 \\ 4 & 2 & -5 & 3 \\ 5 & -7 & -3 & 0 \end{vmatrix} = 2\begin{vmatrix} 1 & -1 & -2 & 0 \\ -3 & 5 & 4 & 5 \\ 4 & 2 & -5 & 3 \\ 5 & -7 & -3 & 0 \end{vmatrix} = 2\begin{vmatrix} 1 & -1 & -2 & 0 \\ 0 & 2 & -2 & 5 \\ 0 & 6 & 3 & 3 \\ 0 & -2 & 7 & 0 \end{vmatrix}$$

$$= 2\begin{vmatrix} 2 & -2 & 5 \\ 6 & 3 & 3 \\ -2 & 7 & 0 \end{vmatrix} = 2\begin{vmatrix} 2 & -2 & 5 \\ 0 & 9 & -12 \\ 0 & 5 & 5 \end{vmatrix}$$

$$= 2\cdot 2\cdot 3\cdot 5\cdot\begin{vmatrix} 3 & -4 \\ 1 & 1 \end{vmatrix} = 2\cdot 2\cdot 3\cdot 5\cdot 7 = 420.$$

   (b) By Cramer's rule express $x_3 = \frac{\det(B)}{\det(A)}$ as a fraction of two determinants. Write down the matrix B in the numerator.

   Sol.

$$B = \begin{bmatrix} 2 & -2 & 3 & 0 \\ -3 & 5 & -2 & 5 \\ 4 & 2 & 1 & 3 \\ 5 & -7 & 0 & 0 \end{bmatrix}$$

   (c) Evaluate $\det(B)$ in the previous problem and find x_3.

   Sol.

$$|B| = \begin{vmatrix} 2 & -2 & 3 & 0 \\ -3 & 5 & -2 & 5 \\ 4 & 2 & 1 & 3 \\ 5 & -7 & 0 & 0 \end{vmatrix} = \begin{vmatrix} -10 & -8 & 3 & -9 \\ 5 & 9 & -2 & 11 \\ 0 & 0 & 1 & 0 \\ 5 & -7 & 0 & 0 \end{vmatrix} = \begin{vmatrix} -10 & -8 & -9 \\ 5 & 9 & 11 \\ 5 & -7 & 0 \end{vmatrix}$$

$$\begin{vmatrix} 0 & 10 & 13 \\ 5 & 9 & 11 \\ 0 & -16 & -11 \end{vmatrix} = -5\begin{vmatrix} 10 & 13 \\ -16 & -11 \end{vmatrix} = -5(10\cdot(-11) - 13\cdot(-16))$$

$$= -490, \quad x_3 = \frac{-490}{420} = -\frac{7}{6}.$$

2. Evaluate the determinant of T.

   Sol.

$$|T| = \begin{vmatrix} a+b+2c & a+b+2c & a+b+2c & a+b+2c \\ b & a & c & c \\ c & c & a & b \\ c & c & b & a \end{vmatrix} = (a+b+2c)\begin{vmatrix} 1 & 1 & 1 & 1 \\ b & a & c & c \\ c & c & a & b \\ c & c & b & a \end{vmatrix}$$

$$= (a+b+2c)\begin{vmatrix} 1 & 0 & 0 & 0 \\ b & a-b & c-b & c-b \\ c & 0 & a-c & b-c \\ c & 0 & b-c & a-c \end{vmatrix} = (a+b+2c)(a-b)((a-c)^2-(b-c)^2)$$

$$= (a-b)^2(a+b+2c)(a+b-2c) = a^4 - 2a^2b^2 + 8abc^2 - 4c^2a^2 + b^4 - 4b^2c^2.$$

Take-Home **Quiz 7** (*Due at 7:00 p.m. on Fri. October 26, 2007*)

Division: **ID#:** **Name:**

Let A be a 5×5 matrix and B a 4×4 matrix given below.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 & x_5 \\ {x_1}^2 & {x_2}^2 & {x_3}^2 & {x_4}^2 & {x_5}^2 \\ {x_1}^3 & {x_2}^3 & {x_3}^3 & {x_4}^3 & {x_5}^3 \\ {x_1}^4 & {x_2}^4 & {x_3}^4 & {x_4}^4 & {x_5}^4 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ {x_1}^2 & {x_2}^2 & {x_3}^2 & {x_4}^2 \\ {x_1}^3 & {x_2}^3 & {x_3}^3 & {x_4}^3 \end{bmatrix}$$

1. Show that $\det(A) = (x_5 - x_1)(x_5 - x_2)(x_5 - x_3)(x_5 - x_4)\det(B)$. (Use the back of this sheet.)

2. Find $\det(A)$. (Solution only.)

3. Let $f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ be a polynomial satisfying $f(-3) = 2$, $f(-1) = 5$, $f(2) = -3$, $f(3) = 0$ and $f(7) = 100$. Write down a system of linear equations to find a_0, a_1, a_2, a_3, a_4 and explain why the numbers a_0, a_1, a_2, a_3, a_4 are uniquely determined. Do not solve the equation!

4. Explain why there are infinitely many polynomials $g(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ satisfying $g(-3) = 2$, $g(-1) = 5$, $g(2) = -3$ and $g(3) = 0$.

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Solutions to Take-Home Quiz 7 (*October 26, 2007*)

Let A be a 5×5 matrix and B a 4×4 matrix given below.

1. Show that $\det(A) = (x_5 - x_1)(x_5 - x_2)(x_5 - x_3)(x_5 - x_4)\det(B)$.

   Sol.

$$
\begin{aligned}
|A| &= \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 & x_5 \\ {x_1}^2 & {x_2}^2 & {x_3}^2 & {x_4}^2 & {x_5}^2 \\ {x_1}^3 & {x_2}^3 & {x_3}^3 & {x_4}^3 & {x_5}^3 \\ {x_1}^4 & {x_2}^4 & {x_3}^4 & {x_4}^4 & {x_5}^4 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & 0 & 1 \\ x_1 - x_5 & x_2 - x_5 & x_3 - x_5 & x_4 - x_5 & x_5 \\ {x_1}^2 - {x_5}^2 & {x_2}^2 - {x_5}^2 & {x_3}^2 - {x_5}^2 & {x_4}^2 - {x_5}^2 & {x_5}^2 \\ {x_1}^3 - {x_5}^3 & {x_2}^3 - {x_5}^3 & {x_3}^3 - {x_5}^3 & {x_4}^3 - {x_5}^3 & {x_5}^3 \\ {x_1}^4 - {x_5}^4 & {x_2}^4 - {x_5}^4 & {x_3}^4 - {x_5}^4 & {x_4}^4 - {x_5}^4 & {x_5}^4 \end{vmatrix} \\
&= (-1)^6 \begin{vmatrix} x_1 - x_5 & x_2 - x_5 & x_3 - x_5 & x_4 - x_5 \\ {x_1}^2 - {x_5}^2 & {x_2}^2 - {x_5}^2 & {x_3}^2 - {x_5}^2 & {x_4}^2 - {x_5}^2 \\ {x_1}^3 - {x_5}^3 & {x_2}^3 - {x_5}^3 & {x_3}^3 - {x_5}^3 & {x_4}^3 - {x_5}^3 \\ {x_1}^4 - {x_5}^4 & {x_2}^4 - {x_5}^4 & {x_3}^4 - {x_5}^4 & {x_4}^4 - {x_5}^4 \end{vmatrix} \quad \text{(Cofactor expansion)} \\
&\overset{[4,3;x_5]}{=} (-1)^6 \begin{vmatrix} x_1 - x_5 & x_2 - x_5 & x_3 - x_5 & x_4 - x_5 \\ {x_1}^2 - {x_5}^2 & {x_2}^2 - {x_5}^2 & {x_3}^2 - {x_5}^2 & {x_4}^2 - {x_5}^2 \\ {x_1}^3 - {x_5}^3 & {x_2}^3 - {x_5}^3 & {x_3}^3 - {x_5}^3 & {x_4}^3 - {x_5}^3 \\ {x_1}^4 - x_5{x_1}^3 & {x_2}^4 - x_5{x_2}^3 & {x_3}^4 - x_5{x_3}^3 & {x_4}^4 - x_5{x_3}^3 \end{vmatrix} \\
&\overset{[3,2;x_5]}{=} (-1)^6 \begin{vmatrix} x_1 - x_5 & x_2 - x_5 & x_3 - x_5 & x_4 - x_5 \\ {x_1}^2 - {x_5}^2 & {x_2}^2 - {x_5}^2 & {x_3}^2 - {x_5}^2 & {x_4}^2 - {x_5}^2 \\ {x_1}^3 - x_5{x_1}^2 & {x_2}^3 - x_5{x_2}^2 & {x_3}^3 - x_5{x_3}^2 & {x_4}^3 - x_5{x_4}^2 \\ {x_1}^4 - x_5{x_1}^3 & {x_2}^4 - x_5{x_2}^3 & {x_3}^4 - x_5{x_3}^3 & {x_4}^4 - x_5{x_3}^3 \end{vmatrix} \\
&\overset{[2,1;x_5]}{=} (-1)^6 \begin{vmatrix} x_1 - x_5 & x_2 - x_5 & x_3 - x_5 & x_4 - x_5 \\ {x_1}^2 - x_5x_1 & {x_2}^2 - x_5x_1 & {x_3}^2 - x_5x_1 & {x_4}^2 - x_5x_1 \\ {x_1}^3 - x_5{x_1}^2 & {x_2}^3 - x_5{x_2}^2 & {x_3}^3 - x_5{x_3}^2 & {x_4}^3 - x_5{x_4}^2 \\ {x_1}^4 - x_5{x_1}^3 & {x_2}^4 - x_5{x_2}^3 & {x_3}^4 - x_5{x_3}^3 & {x_4}^4 - x_5{x_3}^3 \end{vmatrix}
\end{aligned}
$$

Factor out $x_1 - x_5$ from the first column, and $x_2 - x_5$ from the second ...

$$
\begin{aligned}
&= (-1)^6(x_1 - x_5)(x_2 - x_5)(x_3 - x_5)(x_4 - x_5) \begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ {x_1}^2 & {x_2}^2 & {x_3}^2 & {x_4}^2 \\ {x_1}^3 & {x_2}^3 & {x_3}^3 & {x_4}^3 \end{vmatrix} \\
&= (x_5 - x_1)(x_5 - x_2)(x_5 - x_3)(x_5 - x_4)|B|
\end{aligned}
$$

2. Find $\det(A)$. (Solution only.)

   Sol. This is called the Vandermonde determinant. Please be careful on the indices. There are various expression of products just as Σ notation for summations.

$$
\begin{aligned}
|A| &= (x_5 - x_1)(x_5 - x_2)(x_5 - x_3)(x_5 - x_4)(x_4 - x_3)(x_4 - x_2)(x_4 - x_1) \\
&\quad (x_3 - x_2)(x_3 - x_1)(x_2 - x_1) \\
&= \prod_{j=2}^{5}\prod_{i=1}^{j-1}(x_j - x_i) = \prod_{1 \le i < j \le 5}(x_j - x_i) \\
&= (-1)^{10}\prod_{i=1}^{4}\prod_{j=i+1}^{5}(x_i - x_j) = \prod_{1 \le i < j \le 5}^{4}(x_i - x_j).
\end{aligned}
$$

For the general case, the Vandermonde determinant has the following value.

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ {x_1}^2 & {x_2}^2 & {x_3}^2 & \cdots & {x_n}^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ {x_1}^{n-1} & {x_2}^{n-1} & {x_3}^{n-1} & \cdots & {x_n}^{n-1} \end{vmatrix} = \prod_{1\le i<j\le n}(x_j - x_i) = (-1)^{\frac{n(n+1)}{2}} \prod_{1\le i<j\le n}(x_i - x_j).$$

The determinant is nonzero if $x_1, x_2, \ldots, x_n$ are all distinct numbers.

3. Let $f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ be a polynomial satisfying $f(-3) = 2$, $f(-1) = 5$, $f(2) = -3$, $f(3) = 0$ and $f(7) = 100$. Write down a system of linear equations to find a_0, a_1, a_2, a_3, a_4 and explain why the numbers a_0, a_1, a_2, a_3, a_4 are uniquely determined. Do not solve the equation!

Sol.

$$\begin{cases} a_0 + (-3)a_1 + (-3)^2a_2 + (-3)^3a_3 + (-3)^4a_4 &= 2 \\ a_0 + (-1)a_1 + (-1)^2a_2 + (-1)^3a_3 + (-1)^4a_4 &= 5 \\ a_0 + 2a_1 + 2^2a_2 + 2^3a_3 + 2^4a_4 &= -3 \\ a_0 + 3a_1 + 3^2a_2 + 3^3a_3 + 3^4a_4 &= 0 \\ a_0 + 7a_1 + 7^2a_2 + 7^3a_3 + 7^4a_4 &= 100 \end{cases}$$

The coefficient matrix C of this system of linear equation is the transpose of A with $x_1 = -3$, $x_2 = -1$. $x_3 = 2$, $x_4 = 3$ and $x_5 = 7$.

$$C = \begin{vmatrix} 1 & -3 & (-3)^2 & (-3)^3 & (-3)^4 \\ 1 & -1 & (-1)^2 & (-1)^3 & (-1)^4 \\ 1 & 2 & 2^2 & 2^3 & 2^4 \\ 1 & 3 & 2^2 & 3^3 & 3^4 \\ 1 & 7 & 7^2 & 7^3 & 7^4 \end{vmatrix}$$

$$\begin{aligned} |C| &= |C^T| \\ &= (7-3)(7-2)(7-(-1))(7-(-3))(3-2)(3-(-1))(3-(-3)) \\ &\quad (2-(-1))(2-(-3))((-1)-(-3)). \end{aligned}$$

Hence the determinant of the coefficient matrix is nonzero. Therefore a_0, a_1, a_2, a_3, a_4 are uniquely determined.

4. Explain why there are infinitely many polynomials $g(x) = a_4x^4+a_3x^3+a_2x^2+a_1x+a_0$ satisfying $g(-3) = 2$, $g(-1) = 5$, $g(2) = -3$ and $g(3) = 0$.

Sol. By the previous problem for each n with $g(7) = m$, a_0, a_1, a_2, a_3, a_4 are uniquely determined. They are different if m is distinct. Hence there are infinitely many polynomials $g(x)$ satisfying the conditions.

Other Solution. We can find a polynomial with the conditions such that $a_4 = 0$. Let $g(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ be a polynomial satisfying $g(-3) = 2$, $g(-1) = 5$, $g(2) = -3$ and $g(3) = 0$. Then

$$h(x) = g(x) + a_4(x-(-3))(x-(-1))(x-2)(x-3) \quad (a_4 \text{ is any number.})$$

also satisfies $h(-3) = 2$, $h(-1) = 5$, $h(2) = -3$ and $h(3) = 0$. Hence there are infinitely many polynomials of degree 4 satisfying the conditions.

Take-Home **Quiz 8** (*Due at 7:00 p.m. on Fri. November 2, 2007*)

Division: **ID#:** **Name:**

1. Let $\pi = (5, 2, 6, 8, 4, 1, 3, 7)$ be a permutation. Find the number of inversions $\ell(\pi)$ and its sign $\mathrm{sign}(\pi)$.

2. Add missing terms to equate the following.

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{vmatrix} = a_{1,1}a_{2,2}a_{3,3}a_{4,4} - a_{1,1}a_{2,2}a_{3,4}a_{4,3} - a_{1,1}a_{2,3}a_{3,2}a_{4,4} + a_{1,1}a_{2,3}a_{3,4}a_{4,2}$$
$$+a_{1,1}a_{2,4}a_{3,2}a_{4,3} - a_{1,1}a_{2,4}a_{3,3}a_{4,2} - a_{1,2}a_{2,1}a_{3,3}a_{4,4} + a_{1,2}a_{2,1}a_{3,4}a_{4,3} + a_{1,2}a_{2,3}a_{3,1}a_{4,4}$$
$$-a_{1,2}a_{2,3}a_{3,4}a_{4,1} - a_{1,2}a_{2,4}a_{3,1}a_{4,3} + a_{1,2}a_{2,4}a_{3,3}a_{4,1} + a_{1,3}a_{2,1}a_{3,2}a_{4,4} - a_{1,3}a_{2,1}a_{3,4}a_{4,2}$$

3. Find all λ such that $(\lambda I - A)\boldsymbol{x} = \mathbf{0}$ has a nontrivial solution, i.e., $\boldsymbol{x} \neq \mathbf{0}$, where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -18 & 15 & 4 \end{bmatrix}, \ \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \ \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \quad \text{Show work!}$$

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Solutions to Take-Home Quiz 8 *(November 2, 2007)*

1. Let $\pi = (5, 2, 6, 8, 4, 1, 3, 7)$ be a permutation. Find the number of inversions $\ell(\pi)$ and its sign $\text{sign}(\pi)$.

   Sol. $(5,2), (5,4), (5,1), (5,3), (2,1), (6,4), (6,1), (6,3), (8,4), (8,1), (8,3), (8,7), (4,1), (4,3)$ are inversions. Hence $\ell(\pi) = 14$ and $\text{sign}(\pi) = (-1)^{14} = 1$. Therefore π is an even permutation. ∎

2. Add missing terms to equate the following.

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{vmatrix} = a_{1,1}a_{2,2}a_{3,3}a_{4,4} - a_{1,1}a_{2,2}a_{3,4}a_{4,3} - a_{1,1}a_{2,3}a_{3,2}a_{4,4} + a_{1,1}a_{2,3}a_{3,4}a_{4,2}$$

$$+a_{1,1}a_{2,4}a_{3,2}a_{4,3} - a_{1,1}a_{2,4}a_{3,3}a_{4,2} - a_{1,2}a_{2,1}a_{3,3}a_{4,4} + a_{1,2}a_{2,1}a_{3,4}a_{4,3} + a_{1,2}a_{2,3}a_{3,1}a_{4,4}$$

$$-a_{1,2}a_{2,3}a_{3,4}a_{4,1} - a_{1,2}a_{2,4}a_{3,1}a_{4,3} + a_{1,2}a_{2,4}a_{3,3}a_{4,1} + a_{1,3}a_{2,1}a_{3,2}a_{4,4} - a_{1,3}a_{2,1}a_{3,4}a_{4,2}$$

$$-a_{1,3}a_{2,2}a_{3,1}a_{4,4} + a_{1,3}a_{2,2}a_{3,4}a_{4,1} + a_{1,3}a_{2,4}a_{3,1}a_{4,2} - a_{1,3}a_{2,4}a_{3,2}a_{4,1} - a_{1,4}a_{2,1}a_{3,2}a_{4,3}$$

$$+a_{1,4}a_{2,1}a_{3,3}a_{4,2} + a_{1,4}a_{2,2}a_{3,1}a_{4,3} - a_{1,4}a_{2,2}a_{3,3}a_{4,1} - a_{1,4}a_{2,3}a_{3,1}a_{4,2} + a_{1,4}a_{2,3}a_{3,2}a_{4,1}.$$

   The missing permutations and their number of inversions are $\ell(3,2,1,4) = 3$, $\ell(3,2,4,1) = 4$, $\ell(3,4,1,2) = 4$, $\ell(3,4,2,1) = 5$, $\ell(4,1,2,3) = 3$, $\ell(4,1,3,2) = 4$, $\ell(4,2,1,3) = 4$, $\ell(4,2,3,1) = 5$, $\ell(4,3,1,2) = 5$ and $\ell(4,3,2,1) = 6$. Thus the last two lines above are missing terms. ∎

3. Find all λ such that $(\lambda I - A)\boldsymbol{x} = \mathbf{0}$ has a nontrivial solution, i.e., $\boldsymbol{x} \neq \mathbf{0}$, where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -18 & 15 & 4 \end{bmatrix}, \ \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \ \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \quad \text{Show work!}$$

   Sol. $(\lambda I - A)\boldsymbol{x} = \mathbf{0}$ has a nontrivial solution if and only if $\lambda I - A$ is invertible by Theorem 5.1 (1.5.3). Moreover $\lambda I - A$ is invertible if and only if $\det(\lambda I - A) = 0$ by Theorem 8.3 (2.3.3). So we compute $\det(\lambda I - A)$.

$$\begin{aligned} \det(\lambda I - A) &= \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 18 & -15 & \lambda - 4 \end{vmatrix} = \lambda^2(\lambda - 4) + 18 - 15\lambda \\ &= \lambda^3 - 4x\lambda^2 - 15\lambda + 18 = (\lambda - 6)(\lambda - 1)(\lambda + 3). \end{aligned}$$

   Hence $(\lambda I - A)\boldsymbol{x} = \mathbf{0}$ has a nontrivial solution if and only if $\lambda = 6, 1$ or -3. ∎

$$\text{Let } \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \ \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ -3 \\ 9 \end{bmatrix}, \ \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 6 \\ 36 \end{bmatrix}, \text{ and } T = [\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -3 & 6 \\ 1 & 9 & 36 \end{bmatrix}$$

   Then $A\boldsymbol{v}_1 = \boldsymbol{v}_1$, $A\boldsymbol{v}_2 = -3\boldsymbol{v}_2$ and $A\boldsymbol{v}_3 = 6\boldsymbol{v}_3$, and

$$AT = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -18 & 15 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -3 & 6 \\ 1 & 9 & 36 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -3 & 6 \\ 1 & 9 & 36 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 6 \end{bmatrix} = TD,$$

   where D is a diagonal matrix with diagonal entry $1, -3, 6$. T is invertible as its determinant is a Vandermonde type and $T^{-1}AT = D$.