Solutions to Take-Home Quiz 8 (*November 2, 2007*)

1. Let $\pi = (5, 2, 6, 8, 4, 1, 3, 7)$ be a permutation. Find the number of inversions $\ell(\pi)$ and its sign $\operatorname{sign}(\pi)$.

   Sol. $(5, 2), (5, 4), (5, 1), (5, 3), (2, 1), (6, 4), (6, 1), (6, 3), (8, 4), (8, 1), (8, 3), (8, 7), (4, 1), (4, 3)$ are inversions. Hence $\ell(\pi) = 14$ and $\operatorname{sign}(\pi) = (-1)^{14} = 1$. Therefore π is an even permutation. ■

2. Add missing terms to equate the following.

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{vmatrix} = a_{1,1}a_{2,2}a_{3,3}a_{4,4} - a_{1,1}a_{2,2}a_{3,4}a_{4,3} - a_{1,1}a_{2,3}a_{3,2}a_{4,4} + a_{1,1}a_{2,3}a_{3,4}a_{4,2}$$
$$+a_{1,1}a_{2,4}a_{3,2}a_{4,3} - a_{1,1}a_{2,4}a_{3,3}a_{4,2} - a_{1,2}a_{2,1}a_{3,3}a_{4,4} + a_{1,2}a_{2,1}a_{3,4}a_{4,3} + a_{1,2}a_{2,3}a_{3,1}a_{4,4}$$
$$-a_{1,2}a_{2,3}a_{3,4}a_{4,1} - a_{1,2}a_{2,4}a_{3,1}a_{4,3} + a_{1,2}a_{2,4}a_{3,3}a_{4,1} + a_{1,3}a_{2,1}a_{3,2}a_{4,4} - a_{1,3}a_{2,1}a_{3,4}a_{4,2}$$
$$-a_{1,3}a_{2,2}a_{3,1}a_{4,4} + a_{1,3}a_{2,2}a_{3,4}a_{4,1} + a_{1,3}a_{2,4}a_{3,1}a_{4,2} - a_{1,3}a_{2,4}a_{3,2}a_{4,1} - a_{1,4}a_{2,1}a_{3,2}a_{4,3}$$
$$+a_{1,4}a_{2,1}a_{3,3}a_{4,2} + a_{1,4}a_{2,2}a_{3,1}a_{4,3} - a_{1,4}a_{2,2}a_{3,3}a_{4,1} - a_{1,4}a_{2,3}a_{3,1}a_{4,2} + a_{1,4}a_{2,3}a_{3,2}a_{4,1}.$$

   The missing permutations and their number of inversions are $\ell(3, 2, 1, 4) = 3$, $\ell(3, 2, 4, 1) = 4$, $\ell(3, 4, 1, 2) = 4$, $\ell(3, 4, 2, 1) = 5$, $\ell(4, 1, 2, 3) = 3$, $\ell(4, 1, 3, 2) = 4$, $\ell(4, 2, 1, 3) = 4$, $\ell(4, 2, 3, 1) = 5$, $\ell(4, 3, 1, 2) = 5$ and $\ell(4, 3, 2, 1) = 6$. Thus the last two lines above are missing terms. ■

3. Find all λ such that $(\lambda I - A)\boldsymbol{x} = \boldsymbol{0}$ has a nontrivial solution, i.e., $\boldsymbol{x} \neq \boldsymbol{0}$, where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -18 & 15 & 4 \end{bmatrix}, \ \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \ \boldsymbol{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \quad \text{Show work!}$$

   Sol. $(\lambda I - A)\boldsymbol{x} = \boldsymbol{0}$ has a nontrivial solution if and only if $\lambda I - A$ is invertible by Theorem 5.1 (1.5.3). Moreover $\lambda I - A$ is invertible if and only if $\det(\lambda I - A) = 0$ by Theorem 8.3 (2.3.3). So we compute $\det(\lambda I - A)$.

$$\begin{aligned} \det(\lambda I - A) &= \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 18 & -15 & \lambda - 4 \end{vmatrix} = \lambda^2(\lambda - 4) + 18 - 15\lambda \\ &= \lambda^3 - 4x\lambda^2 - 15\lambda + 18 = (\lambda - 6)(\lambda - 1)(\lambda + 3). \end{aligned}$$

   Hence $(\lambda I - A)\boldsymbol{x} = \boldsymbol{0}$ has a nontrivial solution if and only if $\lambda = 6, 1$ or -3. ■

$$\text{Let } \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \ \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ -3 \\ 9 \end{bmatrix}, \ \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 6 \\ 36 \end{bmatrix}, \text{ and } T = [\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -3 & 6 \\ 1 & 9 & 36 \end{bmatrix}$$

Then $A\boldsymbol{v}_1 = \boldsymbol{v}_1$, $A\boldsymbol{v}_2 = -3\boldsymbol{v}_2$ and $A\boldsymbol{v}_3 = 6\boldsymbol{v}_3$, and

$$AT = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -18 & 15 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -3 & 6 \\ 1 & 9 & 36 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -3 & 6 \\ 1 & 9 & 36 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 6 \end{bmatrix} = TD,$$

where D is a diagonal matrix with diagonal entry 1, -3, 6. T is invertible as its determinant is a Vandermonde type and $T^{-1}AT = D$.