Linear Algebra I *November 20, 2014*

Solutions to Final Exam 2014

(Total: 100 pts, 50% of the grade)

1. Let $\boldsymbol{u} = [2, 1, -3]^T$, $\boldsymbol{v} = [0, 1, 2]^T$, $\boldsymbol{w} = [1, 3, 1]^T$, $\boldsymbol{e}_1 = [1, 0, 0]^T$, $\boldsymbol{e}_2 = [0, 1, 0]^T$ and $\boldsymbol{e}_3 = [0, 0, 1]^T$. (10 pts)

   (a) Find $\boldsymbol{u} \times \boldsymbol{v}$ and the volume of the parallelepiped defined by $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$. Show work!
   Solution.

$$\boldsymbol{u} \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ 2 & 1 & -3 \\ 0 & 1 & 2 \end{vmatrix} = \left[\begin{vmatrix} 1 & -3 \\ 1 & 2 \end{vmatrix}, - \begin{vmatrix} 2 & -3 \\ 0 & 2 \end{vmatrix}, \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} \right]^T = \begin{bmatrix} 5 \\ -4 \\ 2 \end{bmatrix}.$$

$$\text{Volume} = |(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w}| = |5 \cdot 1 + (-4) \cdot 3 + 2 \cdot 1| = |-5| = 5.$$

   (b) Find the standard matrix A of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T(\boldsymbol{e}_1) = \boldsymbol{u}$, $T(\boldsymbol{e}_1 + \boldsymbol{e}_2) = \boldsymbol{v}$ and $T(\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3) = \boldsymbol{w}$. Show work!
   Solution.

$$T(\boldsymbol{e}_2) = T(\boldsymbol{e}_1 + \boldsymbol{e}_2) - T(\boldsymbol{e}_1) = \boldsymbol{v} - \boldsymbol{u} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}.$$

$$T(\boldsymbol{e}_3) = T(\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3) - T(\boldsymbol{e}_1 + \boldsymbol{e}_2) = \boldsymbol{w} - \boldsymbol{v} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

   Hence the standard matrix is

$$A = [T(\boldsymbol{e}_1), T(\boldsymbol{e}_2), T(\boldsymbol{e}_3)] = \begin{bmatrix} 2 & -2 & 1 \\ 1 & 0 & 2 \\ -3 & 5 & -1 \end{bmatrix}.$$

2. Consider the system of linear equations with augmented matrix $C = [\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3, \boldsymbol{c}_4, \boldsymbol{c}_5, \boldsymbol{c}_6]$, where $\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_6$ are the columns of C. We obtained a row echelon form G after applying a sequence of elementary row operations to the matrix C. (30 pts)

$$C = \begin{bmatrix} 0 & 0 & 1 & -2 & 0 & -7 \\ 1 & 1 & 0 & 2 & 0 & 9 \\ -1 & -1 & 0 & -1 & -1 & -6 \\ -3 & -3 & -2 & -2 & 0 & -13 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 1 & 0 & 2 & 0 & 9 \\ 0 & 0 & 1 & -2 & 0 & -7 \\ 0 & 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

   (a) Describe each step of a sequence of elementary row operations to obtain G from C by $[i, j]$, $[i, j; c]$, $[i; c]$ notation. Show work.
   Solution.

$$C \xrightarrow{[1,2]} \begin{bmatrix} 1 & 1 & 0 & 2 & 0 & 9 \\ 0 & 0 & 1 & -2 & 0 & -7 \\ -1 & -1 & 0 & -1 & -1 & -6 \\ -3 & -3 & -2 & -2 & 0 & -13 \end{bmatrix} \xrightarrow{[3,1;1]} \begin{bmatrix} 1 & 1 & 0 & 2 & 0 & 9 \\ 0 & 0 & 1 & -2 & 0 & -7 \\ 0 & 0 & 0 & 1 & -1 & 3 \\ -3 & -3 & -2 & -2 & 0 & -13 \end{bmatrix}$$

$$\xrightarrow{[4,1;3]} \begin{bmatrix} 1 & 1 & 0 & 2 & 0 & 9 \\ 0 & 0 & 1 & -2 & 0 & -7 \\ 0 & 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & -2 & 4 & 0 & 14 \end{bmatrix} \xrightarrow{[4,2;2]} \begin{bmatrix} 1 & 1 & 0 & 2 & 0 & 9 \\ 0 & 0 & 1 & -2 & 0 & -7 \\ 0 & 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = G.$$

   Hence the sequence of operations above is $[1, 2]$, $[3, 1; 1]$, $[4, 1; 3]$, $[4, 2; 2]$.

(b) Find an invertible matrix P of size 4 such that $G = PC$ and express P as a product of elementary matrices. Show work.

Solution. P is the matrix obtained by applying the sequence of row operations $[1,2]$, $[3,1;1]$, $[4,1;3]$, $[4,2;2]$ to the identity matrix of size 4 in this order. Hence

$$P = E(4,2;2)E(4,1;3)E(3,1;1)E(1,2) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{bmatrix}.$$

(c) Is P in (b) uniquely determined? Give a brief explanation.

Solution. No. $E(4;2)PC = E(4;2)G = G$ as the fourth row of G is $\mathbf{0}$. Since P is invertible, $E(4;2)P \neq P$. Hence P is not uniquely determined.

(d) Find three columns of C that are linearly <u>independent</u>, and find three columns of C that are linearly <u>dependent</u>. Give a brief explanation.

Solution. $\boldsymbol{c}_1, \boldsymbol{c}_3, \boldsymbol{c}_4$ are linearly independent, as $[\boldsymbol{c}_1, \boldsymbol{c}_3, \boldsymbol{c}_4]$ is row equivalent to the submatrix of G formed by the first, the third and the fourth columns. $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3$ are linearly dependent, as $[\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3]$ is row equivalent to the submatrix of G formed by the first, the second and the third columns.

$$P[\boldsymbol{c}_1, \boldsymbol{c}_3, \boldsymbol{c}_4] = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad P[\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3] = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(e) By applying a sequence of elementary row operations, reduce C to the <u>reduced</u> row echelon form. Show work!

Solution.

$$G \xrightarrow{[1,3;-2]} \begin{bmatrix} 1 & 1 & 0 & 0 & 2 & 3 \\ 0 & 0 & 1 & -2 & 0 & -7 \\ 0 & 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{[2,3;2]} \begin{bmatrix} 1 & 1 & 0 & 0 & 2 & 3 \\ 0 & 0 & 1 & 0 & -2 & -1 \\ 0 & 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(f) Find all solutions of the system of linear equations.

Solution. Let $x_2 = s$ and $x_5 = t$ be free parameters. Then

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -s-2t+3 \\ s \\ 2t-1 \\ t+3 \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -1 \\ 3 \\ 0 \end{bmatrix} + s \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \\ 1 \end{bmatrix}.$$

3. Let A, $\boldsymbol{x}$ and $\boldsymbol{b}$ be a matrix and vectors given below. (20 pts)

$$A = \begin{bmatrix} 4 & -1 & 2 & 0 \\ 1 & 2 & -2 & -1 \\ -1 & -2 & 1 & 1 \\ -2 & 3 & 1 & 2 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

(a) Evaluate $\det(A)$. Show work!

Solution.

$$\det(A) = \begin{vmatrix} 4 & -1 & 2 & 0 \\ 1 & 2 & -2 & -1 \\ -1 & -2 & 1 & 1 \\ -2 & 3 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 4 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & -2 & 1 & 1 \\ 0 & 7 & -1 & 0 \end{vmatrix} = -\begin{vmatrix} 4 & -1 & 2 \\ 0 & 0 & -1 \\ 0 & 7 & -1 \end{vmatrix} = -28.$$

(b) Express y as a quotient (*bun-su*) of determinants when $A\boldsymbol{x} = \boldsymbol{b}$, and write $\operatorname{adj}(A)$, the adjugate of A. Don't evaluate the determinants.

$$y = \frac{\begin{vmatrix} 4 & -1 & 1 & 0 \\ 1 & 2 & 2 & -1 \\ -1 & -2 & 3 & 1 \\ -2 & 3 & 4 & 2 \end{vmatrix}}{\begin{vmatrix} 4 & -1 & 2 & 0 \\ 1 & 2 & -2 & -1 \\ -1 & -2 & 1 & 1 \\ -2 & 3 & 1 & 2 \end{vmatrix}},$$

$$\operatorname{adj}(A) = \begin{bmatrix} \begin{vmatrix} 2 & -2 & -1 \\ -2 & 1 & 1 \\ 3 & 1 & 2 \end{vmatrix}, & -\begin{vmatrix} -1 & 2 & 0 \\ -2 & 1 & 1 \\ 3 & 1 & 2 \end{vmatrix}, & \begin{vmatrix} -1 & 2 & 0 \\ 2 & -2 & -1 \\ 3 & 1 & 2 \end{vmatrix}, & -\begin{vmatrix} -1 & 2 & 0 \\ 2 & -2 & -1 \\ -2 & 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ -2 & 1 & 2 \end{vmatrix}, & \begin{vmatrix} 4 & 2 & 0 \\ -1 & 1 & 1 \\ -2 & 1 & 2 \end{vmatrix}, & -\begin{vmatrix} 4 & 2 & 0 \\ 1 & -2 & -1 \\ -2 & 1 & 2 \end{vmatrix}, & \begin{vmatrix} 4 & 2 & 0 \\ 1 & -2 & -1 \\ -2 & 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 & -1 \\ -1 & -2 & 1 \\ -2 & 3 & 2 \end{vmatrix}, & -\begin{vmatrix} 4 & -1 & 0 \\ -1 & -2 & 1 \\ -2 & 3 & 2 \end{vmatrix}, & \begin{vmatrix} 4 & -1 & 0 \\ 1 & 2 & -1 \\ -2 & 3 & 2 \end{vmatrix}, & -\begin{vmatrix} 4 & -1 & 0 \\ 1 & 2 & -1 \\ -1 & -2 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 2 & -2 \\ -1 & -2 & 1 \\ -2 & 3 & 1 \end{vmatrix}, & \begin{vmatrix} 4 & -1 & 2 \\ -1 & -2 & 1 \\ -2 & 3 & 1 \end{vmatrix}, & -\begin{vmatrix} 4 & -1 & 2 \\ 1 & 2 & -2 \\ -2 & 3 & 1 \end{vmatrix}, & \begin{vmatrix} 4 & -1 & 2 \\ 1 & 2 & -2 \\ -1 & -2 & 1 \end{vmatrix} \end{bmatrix}.$$

4. Let A be the 6×6 matrix given below, where a and b are real numbers. (20 pts)

$$A = \begin{bmatrix} a & b & b & b & b & b \\ b & a & b & b & b & b \\ b & b & a & b & b & b \\ b & b & b & a & b & b \\ b & b & b & b & a & b \\ b & b & b & b & b & a \end{bmatrix}.$$

(a) Find the determinant of A. Show work!

Solution. Since the column sum is $a + 5b$,

$$|A| = \begin{bmatrix} a+5b & a+5b & a+5b & a+5b & a+5b & a+5b \\ b & a & b & b & b & b \\ b & b & a & b & b & b \\ b & b & b & a & b & b \\ b & b & b & b & a & b \\ b & b & b & b & b & a \end{bmatrix} = (a+5b) \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ b & a & b & b & b & b \\ b & b & a & b & b & b \\ b & b & b & a & b & b \\ b & b & b & b & a & b \\ b & b & b & b & b & a \end{bmatrix}$$

$$= (a+5b) \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ b & a-b & 0 & 0 & 0 & 0 \\ b & 0 & a-b & 0 & 0 & 0 \\ b & 0 & 0 & a-b & 0 & 0 \\ b & 0 & 0 & 0 & a-b & 0 \\ b & 0 & 0 & 0 & 0 & a-b \end{bmatrix} = (a+5b)(a-b)^5.$$

(b) Find the characteristic polynomial of A. Give a brief explanation.

Solution. $\det(A - xI)$ is obtained by replacing a by $a - x$. So

$$\det(A - xI) = (a - x + 5b)(a - x - b)^5 = (x - a - 5b)(x - a + b)^5.$$

(c) Find the condition on a and b that the matrix linear transformation $T : \mathbb{R}^6 \to \mathbb{R}^6$ ($\boldsymbol{x} \mapsto A\boldsymbol{x}$) is onto. Give a brief explanation.

Solution. Since A is a square matrix, by the invertible matrix theorem, T is onto if and only if A is invertible. Therefore the condition is $a + 5b \neq 0$ and $a - b \neq 0$, or equivalently $a \neq b, -5b$.

5. Let A be the following matrix. (20 pts)

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 8 \end{bmatrix}.$$

(a) List all eigenvalues of A, and give a reason that A is diagonalizable.

Solution. Eigenvalues of a tridiagonal matrix are diagonal entries, $1, 2, 4, 8$ in this case. Since these eigenvalues are all distinct, A is diagonalizable.

(b) Find an eigenvector of the largest eigenvalue of A. Show work!

Solution. By the previous problem, 8 is the largest eigenvalue.

$$A - 8I = \begin{bmatrix} -7 & 1 & 1 & 1 \\ 0 & -6 & 2 & 2 \\ 0 & 0 & -4 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} -7 & 1 & 1 & 1 \\ 0 & -6 & 2 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} -7 & 1 & 0 & 2 \\ 0 & -6 & 0 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} -7 & 1 & 0 & 2 \\ 0 & 3 & 0 & -2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 21 & -3 & 0 & -6 \\ 0 & 3 & 0 & -2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 21 & 0 & 0 & -8 \\ 0 & 3 & 0 & -2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \boldsymbol{v}_4 = \begin{bmatrix} 8 \\ 14 \\ 21 \\ 21 \end{bmatrix}$$

Hence $\boldsymbol{v}_4$ is an eigenvector for the largest eigenvalue 8.

(c) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. Show work!

Solution.

$$A - 4I = \begin{bmatrix} -3 & 1 & 1 & 1 \\ 0 & -2 & 2 & 2 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix} \to \begin{bmatrix} -3 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 3 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \boldsymbol{v}_3 = \begin{bmatrix} 2 \\ 3 \\ 3 \\ 0 \end{bmatrix}$$

$$A - 2I = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \text{ and } \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Then $A\boldsymbol{v}_1 = \boldsymbol{v}_1, A\boldsymbol{v}_2 = 2\boldsymbol{v}_2, A\boldsymbol{v}_3 = 4\boldsymbol{v}_3, A\boldsymbol{v}_4 = 8\boldsymbol{v}_4$. Therefore

$$P = [\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4] = \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & 3 & 14 \\ 0 & 0 & 3 & 21 \\ 0 & 0 & 0 & 21 \end{bmatrix}, \text{ and } D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}.$$

Note that

$$AP = A[\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4] = [A\boldsymbol{v}_1, A\boldsymbol{v}_2, A\boldsymbol{v}_3, A\boldsymbol{v}_4] = [\boldsymbol{v}_1, 2\boldsymbol{v}_2, 4\boldsymbol{v}_3, 8\boldsymbol{v}_4] = PD.$$

Since $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$ are eigenvectors corresponding to distinct eigenvectors $1, 2, 4, 8$, they are linearly independent and P is invertible.