1. Let $u = [2, 1, -3]^T$, $v = [0, 1, 2]^T$, $w = [1, 3, 1]^T$, $e_1 = [1, 0, 0]^T$, $e_2 = [0, 1, 0]^T$ and $e_3 = [0, 0, 1]^T$.

(a) Find $u \times v$ and the volume of the parallelepiped defined by $u, v, w$. Show work!

(b) Find the standard matrix $A$ of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T(e_1) = u$, $T(e_1 + e_2) = v$ and $T(e_1 + e_2 + e_3) = w$. Show work!

Points:

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<tr>
<th>1.(a)</th>
<th>(b)</th>
<th>2.(a)</th>
<th>(b)</th>
<th>(c)</th>
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<th>3.(a)*</th>
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<tr>
<td>4.(a)*</td>
<td>(b)</td>
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<td>5.(a)</td>
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メッセージ欄：この授業について、特に改善点について、その他何でもどうぞ。
2. Consider the system of linear equations with augmented matrix \( C = [c_1, c_2, c_3, c_4, c_5, c_6] \), where \( c_1, c_2, \ldots, c_6 \) are the columns of \( C \). We obtained a row echelon form \( G \) after applying a sequence of elementary row operations to the matrix \( C \). (30 pts)

\[
C = \begin{bmatrix}
0 & 0 & 1 & -2 & 0 & -7 \\
1 & 1 & 0 & 2 & 0 & 9 \\
-1 & -1 & 0 & -1 & -1 & -6 \\
-3 & -3 & -2 & -2 & 0 & -13
\end{bmatrix}, \quad G = \begin{bmatrix}
1 & 1 & 0 & 2 & 0 & 9 \\
0 & 0 & 1 & -2 & 0 & -7 \\
0 & 0 & 0 & 1 & -1 & 3 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

(a) Describe each step of a sequence of elementary row operations to obtain \( G \) from \( C \) by \([i, j], [i, j; c], [i; c]\) notation. Show work.

(b) Find an invertible matrix \( P \) of size 4 such that \( G = PC \) and express \( P \) as a product of elementary matrices. Show work.

(c) Is \( P \) in (b) uniquely determined? Give a brief explanation.
(d) Find three columns of $C$ that are linearly independent, and find three columns of $C$ that are linearly dependent. Give a brief explanation.

(e) By applying a sequence of elementary row operations, reduce $C$ to the reduced row echelon form. Show work!

(f) Find all solutions of the system of linear equations.
3. Let $A$, $x$ and $b$ be a matrix and vectors given below. (20 pts)

$$A = \begin{bmatrix}
4 & -1 & 2 & 0 \\
1 & 2 & -2 & -1 \\
-1 & -2 & 1 & 1 \\
-2 & 3 & 1 & 2
\end{bmatrix}, \quad x = \begin{bmatrix}
w \\
x \\
y \\
z
\end{bmatrix}, \quad b = \begin{bmatrix}
1 \\
2 \\
3 \\
4
\end{bmatrix}.$$ 

(a) Evaluate $\det(A)$. Show work!

(b) Express $y$ as a quotient ($bun-su$) of determinants when $Ax = b$, and write $\text{adj}(A)$, the adjugate of $A$. Don’t evaluate the determinants.

$$y = \quad , \quad \text{adj}(A) =$$
4. Let $A$ be the $6 \times 6$ matrix given below, where $a$ and $b$ are real numbers. (20 pts)

$$A = \begin{bmatrix}
    a & b & b & b & b & b \\
    b & a & b & b & b & b \\
    b & b & a & b & b & b \\
    b & b & b & a & b & b \\
    b & b & b & a & b & b \\
    b & b & b & b & a & b
\end{bmatrix}. $$

(a) Find the determinant of $A$. Show work!

(b) Find the characteristic polynomial of $A$. Give a brief explanation.

(c) Find the condition on $a$ and $b$ that the matrix linear transformation $T : \mathbb{R}^6 \to \mathbb{R}^6 (\mathbf{x} \mapsto A\mathbf{x})$ is onto. Give a brief explanation.
5. Let $A$ be the following matrix. (20 pts)

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 8 \end{bmatrix}.$$ 

(a) List all eigenvalues of $A$, and give a reason that $A$ is diagonalizable.

(b) Find an eigenvector of the largest eigenvalue of $A$. Show work!

(c) Find an invertible matrix $P$ and a diagonal matrix $D$ such that $P^{-1}AP = D$. Show work!