1. Let $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ be a transformation defined by:

$$T(x_1, x_2, x_3, x_4) = (3x_1 + x_2 - x_4, x_1 + 2x_2 - 3x_3 + 3x_4, -2x_1 + 4x_2 - 2x_3 + 5x_4).$$

(a) Show that $T$ is a linear transformation.

(b) Find the standard matrix $A = [v_1, v_2, v_3, v_4]$ for the linear transformation $T$. 

Points:

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1. Continued from page 1.

(c) Find $v_1 \times v_2$, where $v_1$ and $v_2$ are in (b).

(d) Find the volume of the parallelepiped determined by $v_1, v_2, v_3$, where $v_1, v_2$ and $v_3$ are in (b).

(e) Determine whether $T$ is one-to-one. Explain your answer.

(f) Determine whether $T$ is onto. Explain your answer.
2. Let $A$ be the following $4 \times 4$ matrix and $a, b, c, d$ real numbers. (25 pts)

$$
A = \begin{bmatrix}
1 & x_1 & x_1^2 & x_1^3 \\
1 & x_2 & x_2^2 & x_2^3 \\
1 & x_3 & x_3^2 & x_3^3 \\
1 & x_4 & x_4^2 & x_4^3
\end{bmatrix}.
$$

$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ is called a cubic polynomial.

(a) Show that $\det(A) = (x_2 - x_1)(x_3 - x_1)(x_4 - x_1)(x_3 - x_2)(x_4 - x_2)(x_4 - x_3)$. 
2. Continued from page 3.

(b) Explain that a cubic polynomial \( f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \) is uniquely determined when \( f(1) = 2, f(2) = 0, f(3) = 1, f(4) = 3 \).

(c) Find \( a_3 \) in (b) by Cramer’s rule. Don’t evaluate determinants.

(d) Suppose \( x_1, x_2, x_3, x_4 \) are distinct. Explain that a cubic polynomial \( f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \) is uniquely determined when \( f(x_1) = y_1, f(x_2) = y_2, f(x_3) = y_3, f(x_4) = y_4 \) for any \( y_1, y_2, y_3, y_4 \).
3. Let $A$ and $B$ be matrices given below. (25 pts)

$$A = \begin{bmatrix}
3 & -5 & -5 & -4 & -2 \\
-3 & 4 & 2 & 6 & 6 \\
-3 & 3 & 0 & 6 & 9 \\
-3 & 1 & -4 & 7 & 8 \\
-3 & 6 & 6 & 6 & 7
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & -1 & 0 & -2 & -3 \\
0 & 1 & 2 & 0 & -3 \\
0 & -2 & -5 & 2 & 7 \\
0 & -2 & -4 & 1 & -1 \\
0 & 3 & 6 & 0 & -2
\end{bmatrix}.$$ 

(a) The matrix $B$ is obtained from the matrix $A$ by applying a sequence of elementary row operations. Find (i) such a sequence of elementary row operations, (ii) a matrix $P$ such that $PA = B$, and (iii) $\det(P)$.

(b) Evaluate $\det(A)$. Briefly explain each step.

(c) Write the $(2, 4)$ entry of $\text{adj}(A)$, the adjugate of $A$, as a determinant. Don’t evaluate it.
4. Let \( A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ -1 & 2 & 4 & 1 \\ 1 & -2 & 2 & 5 \end{bmatrix} \). (20 pts)

(a) Explain that \( A \) has eigenvalues 6 and 0 without computing the characteristic polynomial of \( A \).

(b) Find all eigenvalues of \( A \).

(c) Find an invertible matrix \( P \) and a diagonal matrix \( D \) such that \( P^{-1}AP = D \).