Linear Algebra I *November 15, 2012*

Solutions to Final Exam 2012

(Total: 100 pts, 40% of the grade)

1. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a transformation defined by: (30 pts)

$$T(x_1, x_2, x_3) = (3x_1 + x_2, 2x_1 + 2x_2 - 3x_3, -3x_1 + x_2 - 5x_3).$$

(a) Show that T is a linear transformation.

Solution. Let $\boldsymbol{x} = (x_1, x_2, x_3)$, $\boldsymbol{y} = (y_1, y_2, y_3)$. By definition, we need to show that

$$T(\boldsymbol{x} + \boldsymbol{y}) = T(\boldsymbol{x}) + T(\boldsymbol{y}), \quad T(c\boldsymbol{x}) = cT(\boldsymbol{x}).$$

$$\begin{aligned} T(\boldsymbol{x}+\boldsymbol{y}) &= T(x_1+y_1, x_2+y_2, x_3+y_3) \\ &= (3(x_1+y_1)+(x_2+y_2), 2(x_1+y_1)+2(x_2+y_2)-3(x_3+y_3), \\ &\quad -3(x_1+y_1)+(x_2+y_2)-5(x_3+y_3)) \\ &= (3x_1+x_2, 2x_1+2x_2-3x_3, -3x_1+x_2-5x_3) \\ &\quad +(3y_1+y_2, 2y_1+2y_2-3y_3, -3y_1+y_2-5y_3) \\ &= T(\boldsymbol{x})+T(\boldsymbol{y}). \\ T(c\boldsymbol{x}) &= (3cx_1+cx_2, 2cx_1+2cx_2-3cx_3, -3cx_1+cx_2-5cx_3) \\ &= c(3x_1+x_2, 2x_1+2x_2-3x_3, -3x_1+x_2-5x_3) \\ &= cT(\boldsymbol{x}). \end{aligned}$$

(b) Find the standard matrix $A = [\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3]$ for the linear transformation T.

Solution. A satisfies the following:

$$A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_1+x_2 \\ 2x_1+2x_2-3x_3 \\ -3x_1+x_2-5x_3 \end{bmatrix}. \text{ For } \boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \boldsymbol{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$\boldsymbol{v}_1 = [\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3]\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = A\boldsymbol{e}_1 = \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix}, \boldsymbol{v}_2 = A\boldsymbol{e}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \boldsymbol{v}_3 = A\boldsymbol{e}_3 = \begin{bmatrix} 0 \\ -3 \\ -5 \end{bmatrix}.$$

Thus

$$A = [\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3] = \begin{bmatrix} 3 & 1 & 0 \\ 2 & 2 & -3 \\ -3 & 1 & -5 \end{bmatrix}.$$

(c) Find $\boldsymbol{v}_1 \times \boldsymbol{v}_2$, where $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are in (b).

Solution.

$$\boldsymbol{v}_1 \times \boldsymbol{v}_2 = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ 3 & 2 & -3 \\ 1 & 2 & 1 \end{vmatrix} = \left[\begin{vmatrix} 2 & -3 \\ 2 & 1 \end{vmatrix}, -\begin{vmatrix} 3 & -3 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix}\right]^T = \begin{bmatrix} 8 \\ -6 \\ 4 \end{bmatrix}.$$

(d) Let $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3 \in \mathbb{R}^3$. Suppose the volume of the parallelepiped determined by $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$ is 5. What is the volume of the parallelepiped determined by $T(\boldsymbol{u}_1), T(\boldsymbol{u}_2), T(\boldsymbol{u}_3)$. Write a brief explanation.

Solution. Since the volume of the parallelepiped determined by $T(\boldsymbol{u}_1), T(\boldsymbol{u}_2), T(\boldsymbol{u}_3)$ is $|\det(A)|$ times the volume of the parallelepiped determined by $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$, it is

$$|\det(A)| \cdot 5 = \left|\det\begin{bmatrix} 3 & 1 & 0 \\ 2 & 2 & -3 \\ -3 & 1 & -5 \end{bmatrix}\right| \cdot 5 = \left|\begin{vmatrix} 0 & 1 & 0 \\ -4 & 2 & -3 \\ -6 & 1 & -5 \end{vmatrix}\right| \cdot 5 = |-2| \cdot 5 = 10.$$

(e) (i) Show that there is a linear transformation $U : \mathbb{R}^3 \to \mathbb{R}^3$ $(\boldsymbol{x} = (x_1, x_2, x_3) \mapsto U(\boldsymbol{x}) = U(x_1, x_2, x_3)$ such that $U(T(x_1, x_2, x_3)) = (x_1, x_2, x_3)$, i.e., $U(T(\boldsymbol{x})) = \boldsymbol{x}$ and that (ii) the standard matrix of U is A^{-1}.

Solution. Since the determinant of A is nonzero, A is invertible. Let $U : \mathbb{R}^3 \to \mathbb{R}^3$ $(\boldsymbol{x} \mapsto U(\boldsymbol{x}) = A^{-1}\boldsymbol{x}))$. Then $U(T(\boldsymbol{x})) = A^{-1}(A\boldsymbol{x}) = \boldsymbol{x}$. Thus (i) and (ii) hold.

(f) Find the $(2,3)$ entry of A^{-1}.

Solution.

$$(A^{-1})_{2,3} = \frac{1}{|A|} C_{3,2} = \frac{1}{-2}(-1)^{3+2} \begin{vmatrix} 3 & 0 \\ 2 & -3 \end{vmatrix} = -\frac{9}{2}.$$

2. Let A and P be the following 4×4 matrices, and $\boldsymbol{b} \in \mathbb{R}^4$ given below. (20 pts)

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -d & -c & -b & -a \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \\ \lambda^3 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -2 & 3 \\ 1 & 4 & 4 & 9 \\ 1 & 8 & -8 & 27 \end{bmatrix}.$$

(a) Find the characteristic polynomial $p(x) = \det(A - xI)$ of A.

Solution.

$$\begin{aligned} \begin{vmatrix} -x & 1 & 0 & 0 \\ 0 & -x & 1 & 0 \\ 0 & 0 & -x & 1 \\ -d & -c & -b & -a-x \end{vmatrix} &= (-x) \begin{vmatrix} -x & 1 & 0 \\ 0 & -x & 1 \\ -c & -b & -a-x \end{vmatrix} - (-d) \begin{vmatrix} 1 & 0 & 0 \\ -x & 1 & 0 \\ 0 & -x & 1 \end{vmatrix} \\ &= (-x)^2 \begin{vmatrix} -x & 1 \\ -b & -a-x \end{vmatrix} - c(-x) \begin{vmatrix} 1 & 0 \\ x & 1 \end{vmatrix} + d \\ &= x^4 + ax^3 + bx^2 + cx + d. \end{aligned}$$

(b) Show that if λ is an eigenvalue of A, then $\boldsymbol{b}$ is an eigenvector of A corresponding to λ.

Solution. Since $\boldsymbol{b} \neq \boldsymbol{0}$, it suffices to show that $A\boldsymbol{b} = \lambda\boldsymbol{b}$. By assumption λ is an eigenvalue, and hence $0 = p(\lambda) = \lambda^4 + a\lambda^3 + b\lambda^2 + c\lambda + d$. Thus $\lambda^4 = -d - c\lambda - b\lambda^2 - a\lambda^3$. Therefore

$$A\boldsymbol{b} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -d & -c & -b & -a \end{bmatrix} \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \\ \lambda^3 \end{bmatrix} = \begin{bmatrix} \lambda \\ \lambda^2 \\ \lambda^3 \\ -d - c\lambda - b\lambda^2 - a\lambda^3 \end{bmatrix} = \begin{bmatrix} \lambda \\ \lambda^2 \\ \lambda^3 \\ \lambda^4 \end{bmatrix} = \lambda\boldsymbol{b}.$$

(c) Suppose $AP = PD$ for some diagonal matrix D. Determine a, b, c, d and D.

Solution. Let D be a diagonal matrix of size 4 with $1, 2, -2, 3$ on the diagonal. By (b)

$$\begin{aligned} AP &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -d & -c & -b & -a \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -2 & 3 \\ 1 & 2^2 & (-2)^2 & 3^2 \\ 1^3 & 2^3 & (-2)^3 & 3^3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -2 & 3 \\ 1 & 2^2 & (-2)^2 & 3^2 \\ 1^3 & 2^3 & (-2)^3 & 3^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} = PD. \end{aligned}$$

Now the characteristic polynomial of A equals the characteristic polynomial of D and

$p(x) = (1-x)(2-x)(-2-x)(3-x) = (x-1)(x-2)(x+2)(x-3) = x^4-4x^3-x^2+16x-12.$

Therefore $a = -4, b = -1, c = 16, d = -12$.

3. Let A, B, $\boldsymbol{x}$ and $\boldsymbol{b}$ be matrices and vectors given below. Assume $A\boldsymbol{x} = \boldsymbol{b}$. (20 pts)

$$A = \begin{bmatrix} 0 & 2 & 1 & 3 & 4 \\ -2 & 2 & -3 & -2 & 2 \\ 0 & -2 & -4 & 3 & 1 \\ -3 & 3 & 1 & -7 & -2 \\ 1 & -1 & 2 & 3 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 4 & 2 \\ 0 & -2 & -4 & 3 & 1 \\ 0 & 0 & 7 & 2 & -2 \\ 0 & 2 & 1 & 3 & 4 \end{bmatrix}, \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 3 \\ 1 \\ 4 \\ 1 \\ 5 \end{bmatrix}.$$

(a) The matrix B is obtained from the matrix A by applying a sequence of elementary row operations. (i) Find a matrix P such that $PA = B$, and (ii) express P as a product of elementary matrixes $E(i;c)$, $E(i,j)$, $E(i,j;c)$.
Solution. One of the sequence is $[1,5] \to [2,1;2] \to [4,1;3]$. Hence

$$\begin{aligned} P &= E(4,1;3)E(2,1;2)E(1,5) \\ &= \begin{bmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 3&0&0&1&0 \\ 0&0&0&0&1 \end{bmatrix} \begin{bmatrix} 1&0&0&0&0 \\ 2&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{bmatrix} \begin{bmatrix} 0&0&0&0&1 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 1&0&0&0&0 \end{bmatrix} = \begin{bmatrix} 0&0&0&0&1 \\ 0&1&0&0&2 \\ 0&0&1&0&0 \\ 0&0&0&1&3 \\ 1&0&0&0&0 \end{bmatrix} \end{aligned}$$

(b) Evaluate $\det(A)$. Briefly explain each step.
Solution. Using the fact that $|E(i;c)X| = c|X|$, $|E(i,j)X| = -|X|$, $|E(i,j;c)X| = |X|$ and cofactor expansions,

$$\begin{aligned} |A| &= -|B| = -\begin{vmatrix} 0 & 1 & 4 & 2 \\ -2 & -4 & 3 & 1 \\ 0 & 7 & 2 & -2 \\ 2 & 1 & 3 & 4 \end{vmatrix} = -\begin{vmatrix} 0 & 1 & 4 & 2 \\ 0 & -3 & 6 & 5 \\ 0 & 7 & 2 & -2 \\ 2 & 1 & 3 & 4 \end{vmatrix} = 2\begin{vmatrix} 1 & 4 & 2 \\ -3 & 6 & 5 \\ 7 & 2 & -2 \end{vmatrix} \\ &= 2\begin{vmatrix} -13 & 0 & 6 \\ -24 & 0 & 11 \\ 7 & 2 & -2 \end{vmatrix} = -4\begin{vmatrix} -13 & 6 \\ -24 & 11 \end{vmatrix} = -4\begin{vmatrix} -13 & 6 \\ 2 & -1 \end{vmatrix} = -4. \end{aligned}$$

(c) The matrix P in (a) is uniquely determined. Give your reason.
Solution. Since $\det(A) \neq 0$, A is invertible and A^{-1} is uniquely determined. Hence $P = PAA^{-1} = BA^{-1}$.

(d) Applying the Cramer's rule and express x_2 and x_5 as quotients of determinants. Do not evaluate determinants.
Solution. By (b) $\det(A) = -4$. Now by Cramer's rule,

$$x_2 = \frac{1}{|A|}\begin{vmatrix} 0 & 3 & 1 & 3 & 4 \\ -2 & 1 & -3 & -2 & 2 \\ 0 & 4 & -4 & 3 & 1 \\ -3 & 1 & 1 & -7 & -2 \\ 1 & 5 & 2 & 3 & 0 \end{vmatrix}, \quad x_5 = \frac{1}{|A|}\begin{vmatrix} 0 & 2 & 1 & 3 & 3 \\ -2 & 2 & -3 & -2 & 1 \\ 0 & -2 & -4 & 3 & 4 \\ -3 & 3 & 1 & -7 & 1 \\ 1 & -1 & 2 & 3 & 5 \end{vmatrix}.$$

4. Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 12 & 6 & 4 \\ 0 & 5 & 8 \end{bmatrix}$. (30 pts)

(a) Show that the characteristic polynomial of A is equal to the characteristic polynomial of A^T.

Solution. Since the determinant of a square matrix is equal to the determinant of the transpose of the matrix,

$$\det(A - xI) = \det((A - xI)^T) = \det(A^T - xI^T) = \det(A^T - xI).$$

Thus the characteristic polynomial of A is equal to the characteristic polynomial of A^T.

(b) Show that 12 is an eigenvalue of A.

Solution. Since

$$A^T \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 12 & 0 \\ 1 & 6 & 5 \\ 0 & 4 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 12 \\ 12 \\ 12 \end{bmatrix} = 12 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

and the vector $[1,1,1]^T$ is nonzero, 12 is an eigenvector of A^T. Thus 12 is an eigenvalue of A as well by (a).

(c) Find an eigenvector of A corresponding to an eigenvalue 12.

Solution.

$$A - 12I = \begin{bmatrix} -12 & 1 & 0 \\ 12 & 6-12 & 4 \\ 0 & 5 & 8-12 \end{bmatrix} = \begin{bmatrix} -12 & 1 & 0 \\ 0 & -5 & 4 \\ 0 & 5 & -4 \end{bmatrix} = \begin{bmatrix} -12 & 1 & 0 \\ 0 & -5 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore $[1, 12, 15]^T$ is an eigenvector of 12 for A.

(d) Find all eigenvalues of A.

Solution.

$$\begin{aligned} |A - xI| &= \begin{vmatrix} -x & 1 & 0 \\ 12 & 6-x & 4 \\ 0 & 5 & 8-x \end{vmatrix} = \begin{vmatrix} -x+12 & -x+12 & -x+12 \\ 12 & 6-x & 4 \\ 0 & 5 & 8-x \end{vmatrix} \\ &= (12-x) \begin{vmatrix} 1 & 1 & 1 \\ 12 & 6-x & 4 \\ 0 & 5 & 8-x \end{vmatrix} = (12-x) \begin{vmatrix} 1 & 1 & 1 \\ 0 & -6-x & -8 \\ 0 & 5 & 8-x \end{vmatrix} \\ &= (12-x)(-(6+x)(8-x)+40) = (12-x)(x^2-2x-8) \\ &= -(x-12)(x-4)(x+2). \end{aligned}$$

Therefore eigenvalues are $12, 4, -2$.

(e) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

Solution. Eivenvectors for eigenvalues 4 and -2 are as follows.

$$A - 4I = \begin{bmatrix} -4 & 1 & 0 \\ 12 & 6-4 & 4 \\ 0 & 5 & 8-4 \end{bmatrix} = \begin{bmatrix} -4 & 1 & 0 \\ 0 & 5 & 4 \\ 0 & 5 & 4 \end{bmatrix} = \begin{bmatrix} -4 & 1 & 0 \\ 0 & 5 & 4 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore $[1, 4, -5]^T$ is an eigenvector of 4.

$$A + 2I = \begin{bmatrix} 2 & 1 & 0 \\ 12 & 6+2 & 4 \\ 0 & 5 & 8+2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 4 \\ 0 & 5 & 10 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore $[1, -2, 1]^T$ is an eigenvector of -2. Thus

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 12 & 4 & -2 \\ 15 & -5 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 12 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$