Introduction to Linear Algebra *November 17, 2011*

Solutions to Final Exam 2011 (Total: 100 pts)

1. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by: (15 pts)

$$T(x_1, x_2, x_3) = (2x_1 + 3x_2 + x_3, x_1 - x_2 - x_3, -x_1 + 3x_2 + 5x_3).$$

Let $A = [\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3]$ be the standard matrix for the linear transformation T.

(a) Find A. Show work!
Solution. A satisfies the following:

$$A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 + x_3 \\ x_1 - x_2 - x_3 \\ -x_1 + 3x_2 + 5x_3 \end{bmatrix}. \text{ For } \boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \boldsymbol{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$\boldsymbol{v}_1 = [\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3]\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = A\boldsymbol{e}_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \boldsymbol{v}_2 = A\boldsymbol{e}_2 = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}, \boldsymbol{v}_3 = A\boldsymbol{e}_3 = \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}.$$

Thus

$$A = [\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3] = \begin{bmatrix} 2 & 3 & 1 \\ 1 & -1 & -1 \\ -1 & 3 & 5 \end{bmatrix}.$$

(b) Find $\boldsymbol{v}_1 \times \boldsymbol{v}_2$. Show work!
Solution.

$$\boldsymbol{v}_1 \times \boldsymbol{v}_2 = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ 2 & 1 & -1 \\ 3 & -1 & 3 \end{vmatrix} = \left[\begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix}, \begin{vmatrix} -1 & 2 \\ 3 & 3 \end{vmatrix}, \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} \right]^T = \begin{bmatrix} 2 \\ -9 \\ -5 \end{bmatrix}.$$

(c) Let $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3 \in \mathbb{R}^3$. Suppose the volume of the parallelepiped determined by $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$ is 2. What is the volume of the parallelepiped determined by $T(\boldsymbol{u}_1), T(\boldsymbol{u}_2), T(\boldsymbol{u}_3)$. Write a brief explanation.
Solution. Since the volume of the parallelepiped determined by $T(\boldsymbol{u}_1), T(\boldsymbol{u}_2), T(\boldsymbol{u}_3)$ is $|\det(A)|$ times the volume of the parallelepiped determined by $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$, it is

$$|\det(A)| \cdot 2 = \left| \begin{vmatrix} 2 & 3 & 1 \\ 1 & -1 & -1 \\ -1 & 3 & 5 \end{vmatrix} \right| \cdot 2 = \left| \begin{vmatrix} 0 & 5 & 3 \\ 1 & -1 & -1 \\ 0 & 2 & 4 \end{vmatrix} \right| \cdot 2 = |-14| \cdot 2 = 28.$$

2. Let A and B be the following 4×4 matrices, and $\boldsymbol{b} \in \mathbb{R}^4$ given below. (30 pts)

$$A = \begin{bmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{bmatrix}, \quad B = \begin{bmatrix} b & b & b & b \\ b & b & b & b \\ b & b & b & b \\ b & b & b & b \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

(a) Find the determinant of A. Show work!

Solution. First apply $[1,2;1],[1,3;1],[1,4;1]$ to columns, eg. add one times the second column to the first, etc,. we have

$$\begin{vmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{vmatrix} = \begin{vmatrix} a+3b & b & b & b \\ a+3b & a & b & b \\ a+3b & b & a & b \\ a+3b & b & b & a \end{vmatrix} = (a+3b)\begin{vmatrix} 1 & b & b & b \\ 1 & a & b & b \\ 1 & b & a & b \\ 1 & b & b & a \end{vmatrix}$$

$$= (a+3b)\begin{vmatrix} 1 & b & b & b \\ 0 & a-b & 0 & 0 \\ 0 & 0 & a-b & 0 \\ 0 & 0 & 0 & a-b \end{vmatrix} = (a+3b)(a-b)^3.$$

(b) Find the condition that the set of columns of A is linearly <u>dependent</u>. Write a brief explanation.

Solution. The set of columns of A is linearly dependent if and only if $A\boldsymbol{x} = \mathbf{0}$ has a nontrivial solution. The latter condition is equivalent to A being singular (not invertible). Thus the condition is equivalent to $\det(A) = 0$. Therefore $a + 3b = 0$ or $a = b$.

(c) Find the eigenvalues of A, and their multiplicities.

Solution. Let $f(x)$ be the characteristic polynomial of A. Then

$$f(x) = \det(A - xI) = \begin{vmatrix} a-x & b & b & b \\ b & a-x & b & b \\ b & b & a-x & b \\ b & b & b & a-x \end{vmatrix} = ((a-x)+3b)(a-x-b)^3$$

by replacing a by $a - x$ in the previous problem. Hence

$$f(x) = (x - (a+3b))(x - (a-b))^3,$$

and $a + 3b$ is an eigenvalue with multiplicity 1 and $a - b$ with multiplicity 3 unless $a+3b = a-b$, in which case A has only one eigenvalue a with multiplicity 4 and $b = 0$.

(d) Suppose $b \neq 0$. Find a linearly independent set of vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ such that $B\boldsymbol{v}_1 = B\boldsymbol{v}_2 = B\boldsymbol{v}_3 = \mathbf{0}$. Show that it is actually linearly independent.

Solution. By solving the equation $B\boldsymbol{x} = \mathbf{0}$ by augmented matrix, we have

$$\begin{bmatrix} b & b & b & b & 0 \\ b & b & b & b & 0 \\ b & b & b & b & 0 \\ b & b & b & b & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s\cdot\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\cdot\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u\cdot\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Let $\boldsymbol{v}_1 = [-1,1,0,0]^T$, $\boldsymbol{v}_2 = [-1,0,1,0]^T$ and $\boldsymbol{v}_3 = [-1,0,0,1]^T$. If $s\boldsymbol{v}_1 + t\boldsymbol{v}_2 + u\boldsymbol{v}_2 = \mathbf{0}$, then $[-s-t-u, s, t, u] = [0,0,0,0]$. In this case we must have $s = t = u = 0$. Therefore the set of vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is linearly independent.

(e) Let $T = [\boldsymbol{b}, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3]$. (i) Show that T is invertible, and (ii) there is a diagonal matrix D such that $T^{-1}AT = D$.

Solution. In our case above,

$$T = \begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \text{ and } \det(T) = \begin{vmatrix} 4 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 4 \neq 0.$$

Hence T is invertible.

$$AT = A[\boldsymbol{b}, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3] = [(a+3b)\boldsymbol{b}, (a-b)\boldsymbol{v}_1, (a-b)\boldsymbol{v}_2, (a-b)\boldsymbol{v}_3] = TD,$$

where D is a diagonal matrix with $a+3b, a-b, a-b, a-b$ on the diagonal. Since T is invertible, $T^{-1}AT = D$.

Remarks. First note that $B\boldsymbol{b} = (4b)\cdot\boldsymbol{b}$. So $BT = B[\boldsymbol{b}, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3] = [B\boldsymbol{b}, B\boldsymbol{v}_1, B\boldsymbol{v}_2, B\boldsymbol{v}_3] = [4b\boldsymbol{b}, \boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0}] = [\boldsymbol{b}, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3]\text{diag}(4b, 0, 0, 0) = T\text{diag}(4b, 0, 0, 0)$, where $\text{diag}(4b, 0, 0, 0)$ is a diagonal matrix with $4b, 0, 0, 0$ on its diagonal. Thus B is diagonalized by T. Note that $4b \neq 0$, $\boldsymbol{b}$ cannot be written as a linear combination of $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$. Thus the columns of T are linearly independent and T is invertible. Since $A = (a-b)I + B$, we are done. Check the following.

$$T^{-1}AT = T^{-1}((a-b)I+B)T = (a-b)I+T^{-1}BT = (a-b)I+\text{diag}(4b, 0, 0, 0) = D.$$

3. Let A, $\boldsymbol{x}$ and $\boldsymbol{b}$ be a matrix and vectors given below. Assume $A\boldsymbol{x} = \boldsymbol{b}$. (35 pts)

$$A = \begin{bmatrix} 3 & -1 & 2 & 0 \\ 1 & 2 & -2 & 5 \\ -1 & 3 & 1 & 1 \\ 2 & 3 & 1 & -2 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

(a) Evaluate $\det(A)$. Briefly explain each step.

Solution. The first 4 steps are exactly the same as in (d). Then $[3, 2, -5], [4, 2; 7]$ and we have the 2×2 determinant by expanding along the first and the second column. The rest are easy.

$$A = \begin{vmatrix} 3 & -1 & 2 & 0 \\ 1 & 2 & -2 & 5 \\ -1 & 3 & 1 & 1 \\ 2 & 3 & 1 & -2 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & -2 & 5 \\ 3 & -1 & 2 & 0 \\ -1 & 3 & 1 & 1 \\ 2 & 3 & 1 & -2 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & -2 & 5 \\ 0 & -7 & 8 & -15 \\ 0 & 5 & -1 & 6 \\ 0 & -1 & 5 & -12 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & -2 & 5 \\ 0 & -1 & 5 & -12 \\ 0 & 5 & -1 & 6 \\ 0 & -7 & 8 & -15 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & -2 & 5 \\ 0 & 1 & -5 & 12 \\ 0 & 5 & -1 & 6 \\ 0 & -7 & 8 & -15 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & -2 & 5 \\ 0 & 1 & -5 & 12 \\ 0 & 0 & 24 & -54 \\ 0 & 0 & -27 & 69 \end{vmatrix}$$

$$= -\begin{vmatrix} 24 & -54 \\ -27 & 69 \end{vmatrix} = -\begin{vmatrix} 24 & -54 \\ -3 & 15 \end{vmatrix} = -6\cdot 3\begin{vmatrix} 4 & -9 \\ -1 & 5 \end{vmatrix} = -198.$$

(b) Applying the Cramer's rule and express x_2 and x_3 as quotients of determinants. Do not evaluate determinants.

Solution.

$$x_2 = \frac{\det(A_2)}{\det(A)}, x_3 = \frac{\det(A_3)}{\det(A)} \text{ with } A_2 = \begin{bmatrix} 3 & 1 & 2 & 0 \\ 1 & 2 & -2 & 5 \\ -1 & 3 & 1 & 1 \\ 2 & 4 & 1 & -2 \end{bmatrix}, A_3 = \begin{bmatrix} 3 & -1 & 1 & 0 \\ 1 & 2 & 2 & 5 \\ -1 & 3 & 3 & 1 \\ 2 & 3 & 4 & -2 \end{bmatrix}.$$

(c) Let $B = \text{adj}(A)$, the adjugate of A. Determine the $(2,3)$-entry of B .
Do not evaluate the determinant involved.

Solution.

$$B_{2,3} = (-1)^{2+3} \begin{vmatrix} 3 & 2 & 0 \\ 1 & -2 & 5 \\ 2 & 1 & -2 \end{vmatrix} = - \begin{vmatrix} 3 & 2 & 0 \\ 1 & -2 & 5 \\ 2 & 1 & -2 \end{vmatrix}.$$

Let B be the augmented matrix of $A\boldsymbol{x} = \boldsymbol{b}$. Let C be a matrix obtained from B after applying a series of elementary row operations.

$$B = \begin{bmatrix} 3 & -1 & 2 & 0 & 1 \\ 1 & 2 & -2 & 5 & 2 \\ -1 & 3 & 1 & 1 & 3 \\ 2 & 3 & 1 & -2 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & -2 & 5 & 2 \\ 0 & 1 & -5 & 12 & 0 \\ 0 & 5 & -1 & 6 & 5 \\ 0 & -7 & 8 & -15 & -5 \end{bmatrix}.$$

(d) Write a sequence of operations applied to B to obtain C using $[i;c]$, $[i,j]$, $[i,j;c]$ notation.

Solution. $[1,2] \to [2,1;-3] \to [3,1;1] \to [4,1;-2] \to [2,4] \to [2;-1]$.

(e) Find a 4×4 matrix P such that $PB = C$.

Solution.

$$P = E(2;-1)E(2,4)E(4,1;-2)E(3,1;1)E(2,1;-3)E(1,2) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 1 & -3 & 0 & 0 \end{bmatrix}.$$

(f) Express P^{-1} as a product of elementary matrices using the notation $E(i;c)$, $E(i,j)$, $E(i,j;c)$.

Solution.

$$\begin{aligned} P^{-1} &= (E(2;-1)E(2,4)E(4,1;-2)E(3,1;1)E(2,1;-3)E(1,2))^{-1} \\ &= E(1,2)E(2,1;3)E(3,1;-1)E(4,1;2)E(2,4)E(2;-1). \end{aligned}$$

(g) Explain that P in (e) is uniquely determined.

Solution. Let C' be the 4×4 matrix consisting of the first 4 columns of C. Since the first four columns of B forms A, we have $PA = C'$. Since $\det(A) \neq 0$ in (a), A is invertible. Thus $P = C'A^{-1}$ and P is uniquely determined.

4. Let $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$, $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_3 \in \mathbb{R}^3$ be as follows. (20 pts)

$$\boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \boldsymbol{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \boldsymbol{v}_3 = \begin{bmatrix} 2 \\ 3 \\ 8 \end{bmatrix}.$$

(a) Find the reduced row echelon form of the following matrix. (Show work.)

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 4 & 1 & 8 & 0 & 0 & 1 \end{bmatrix}$$

Solution.

$$\left[\begin{array}{rrrrrr} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 4 & 1 & 8 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrrrrr} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & -1 & -1 & -2 & 1 & 0 \end{array}\right] \to \left[\begin{array}{rrrrrr} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & -1 & -6 & 1 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{rrrrrr} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right] \to \left[\begin{array}{rrrrrr} 1 & 0 & 0 & -11 & 2 & 2 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right].$$

(b) Using (a), explain that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is linearly independent.

Solution. Let $B = [\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3]$. Then $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is linearly independent if and only if B is invertible. B is invertible if and only if its reduced echelon form is I. This is the case as we have seen above. Hence $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is linearly independent.

(c) Find the standard matrix A of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T(\boldsymbol{v}_1) = \boldsymbol{e}_1$, $T(\boldsymbol{v}_2) = \boldsymbol{e}_2$ and $T(\boldsymbol{v}_3) = \boldsymbol{e}_3$.

Solution. As in the previous problem let $B = [\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3]$. Then by out assumption, we have

$$AB = A[\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3] = [A\boldsymbol{v}_1, A\boldsymbol{v}_2, A\boldsymbol{v}_3] = [\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3] = I.$$

Thus B is invertible and $A = B^{-1}$ and by (a)

$$B = \left[\begin{array}{rrr} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{array}\right].$$