Introduction to Linear Algebra

November 18, 2010

Final Exam 2010

(Total: 100 pts)

Division: ID#: Name:

1. Let \( u, v, w \) be as follows. (10 pts)

\[
\mathbf{u} = (4, -8, 1), \quad \mathbf{v} = (2, 1, -2), \quad \mathbf{w} = (3, -4, 12).
\]

(a) The vector \( \mathbf{p} = \text{proj}_v \mathbf{u} \) is a scalar multiple of \( \mathbf{v} \) such that \( \mathbf{u} - \mathbf{p} \) is orthogonal to \( \mathbf{v} \). Find \( \mathbf{p} \). (Show work.)

(b) Compute \( \mathbf{u} \times \mathbf{v} \), and find the volume of the parallelepiped (heiko-6-mentai) in 3-space determined by the vectors \( \mathbf{u}, \mathbf{v}, \mathbf{w} \). (Show work.)

Points:

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* 10pts else 5 pts
2. Evaluate the following determinant. Write explanation in words in detail at each step. (10 pts)

\[
\begin{vmatrix}
\lambda - c_1 & -c_2 & \cdots & -c_n \\
-c_1 & \lambda - c_2 & \cdots & -c_n \\
\vdots & \vdots & \ddots & \vdots \\
-c_1 & -c_2 & \cdots & \lambda - c_n \\
\end{vmatrix}
= 
\]
3. Let $A$, $x$ and $b$ be the matrices below. Assume $Ax = b$.

$$
A = \begin{bmatrix}
1 & 0 & 2 & 3 \\
-2 & 1 & 4 & 12 \\
2 & -2 & 1 & 1 \\
2 & 1 & 1 & -3
\end{bmatrix},
\quad
x = \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix},
\quad
b = \begin{bmatrix}
1 \\
2 \\
-2 \\
3
\end{bmatrix}.
$$

(a) Evaluate $\det(A)$. Briefly explain each step. (10 pts)

(b) Applying the Cramer’s rule and express $x_1$ and $x_4$ as quotients of determinants. Do not evaluate determinants. (5 pts)
(c) Explain that the following system of linear equations with unknowns $y_1, y_2, y_3, y_4, y_5, y_6$ is always consistent and the solution can be written with two free parameters for any $a, b, c, d, e, f, g$ and $h$. (5 pts)

$$
\begin{cases}
y_1 + 2y_3 + 3y_4 + ay_5 + ey_6 = 1 \\
-2y_1 + y_2 + 4y_3 + 12y_4 + by_5 + fy_6 = 2 \\
2y_1 - 2y_2 + y_3 + y_4 + cy_5 + gy_6 = -2 \\
2y_1 + y_2 + y_3 - 3y_4 + dy_5 + hy_6 = 3
\end{cases}
$$

(d) Let $H$ and $H'$ be as below. Suppose $A^{-1}H = H'$. Explain that the solutions to the system of linear equations in (c) can be expressed as follows, where $s$ and $t$ are free parameters. (5 pts)

$$
H = \begin{bmatrix} a & e \\ b & f \\ c & g \\ d & h \end{bmatrix}, \quad H' = \begin{bmatrix} a' & e' \\ b' & f' \\ c' & g' \\ d' & h' \end{bmatrix}, \quad \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ 0 \\ 0 \end{bmatrix} - s \cdot \begin{bmatrix} a' \\ b' \\ c' \\ d' \\ 0 \\ -1 \end{bmatrix} - t \cdot \begin{bmatrix} e' \\ f' \\ g' \\ h' \\ 0 \\ -1 \end{bmatrix}.
$$
4. Let $B$ be the augmented matrix of a system of linear equations. Let $C$ be a matrix obtained from $B$ after a series of elementary row operation. (25 pts)

\[
B = \begin{bmatrix}
1 & 0 & 2 & 3 & a \\
-2 & 1 & 4 & 12 & b \\
2 & -2 & 1 & 1 & c \\
2 & 1 & 1 & -3 & d
\end{bmatrix}, \quad C = \begin{bmatrix}
1 & 0 & 2 & 3 & a' \\
0 & 1 & -3 & -9 & b' \\
0 & -2 & -3 & -5 & c' \\
0 & 1 & 8 & 18 & d'
\end{bmatrix}.
\]

(a) Express $a'$, $b'$, $c'$ and $d'$ in terms of $a$, $b$, $c$, $d$. (Show work.)

(b) Write the sequence of operations applied to $B$ to obtain $C$ using $[i; c]$, $[i, j]$, $[i, j; c]$ notation.

(c) Let $P$ be a $4 \times 4$ matrix such that $PB = C$. Express each of $P$ and $P^{-1}$ as a product of elementary matrices using the notation $P(i; c)$, $P(i, j)$, $P(i, j; c)$.

(d) Determine $P$ and $P^{-1}$. (Solution only.)

(e) Explain that $P$ in (c) is uniquely determined.
5. Let $A$, $\mathbf{x}$, $\mathbf{b}_n$ ($n = 0, 1, 2, \ldots$) be as follows. (30 pts)

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & 5 & 2 \end{bmatrix}, \quad \mathbf{b}_n = \begin{bmatrix} a_n \\ a_{n+1} \\ a_{n+2} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_{n+1} \\ a_{n+2} \\ a_{n+3} \end{bmatrix} = \mathbf{b}_{n+1} = A\mathbf{b}_n = A \begin{bmatrix} a_n \\ a_{n+1} \\ a_{n+2} \end{bmatrix}.$$

(a) Find the cofactor matrix $\tilde{A}$, the adjoint matrix $\text{adj}(A)$ and the inverse of $A$. (Solution only.)

(b) Find the characteristic polynomial and the eigenvalues of $A$. (Show work.)

(c) Find an eigenvector corresponding to each of the eigenvalues of $A$. (Show work.)
(d) Find a $3 \times 3$ matrix $P$ and a diagonal matrix $D$ such that $AP = PD$. (Give explanation.)

(e) When $a_0 = 1$, $a_1 = -4$ and $a_2 = -4$, find $a_n$. (Show work.)