BCM I : Final 2018

June 20, 2018

ID#: Name:

1. Let P, Q, R be statements. (5 pts x 2 = 10 pts)

   (a) Complete the following truth table.

P	Q	R	$(P \vee \sim Q) \Rightarrow R$	$(\sim P \vee R) \wedge (Q \vee R)$
T	T	T		
T	T	F		
T	F	T		
T	F	F		
F	T	T		
F	T	F		
F	F	T		
F	F	F		

   (b) Show $(P \vee \sim Q) \Rightarrow R \equiv (\sim P \vee R) \wedge (Q \vee R)$ by using formulas.

2. Show that there is an integer m such that for each integer $n \geq m$, there are <u>positive</u> integers a and b such that $n = 4a + 5b$. (10 pts)

1. (10)	2. (10)	3. (20)	4. (20)	5. (20)	6. (20)	Total

ID#: Name:

3. Let p be a prime number, let x, y and z be integers such that $x^2 + y^2 = pz^2$. Prove or disprove each of the following statements. (5 pts x 4 = 20 pts)

   (a) If p divides both x and y, then p divides z.

   (b) If p does not divide y, there exists an integer w such that $w^2 + 1 \equiv 0 \pmod{p}$.

   (c) If $p = 5$, i.e., $x^2 + y^2 = 5z^2$, then $x = y = z = 0$.

   (d) If $p = 7$, i.e., $x^2 + y^2 = 7z^2$, then $x = y = z = 0$.

ID#: Name:

4. Let $f : X \to Y$, $g : Y \to Z$ and $h = g \circ f : X \to Z$ ($x \mapsto g(f(x))$) be functions. Prove or disprove the following. (5 pts x 4 = 20 pts)

   (a) If f is onto and g is onto, then h is onto.

   (b) If h is one-to-one, then g is one-to-one.

   (c) If h is one-to-one, then f is one-to-one.

   (d) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ for all subsets A, B in Y.

ID#: Name:

5. For $a, b \in \boldsymbol{R}$ with $a < b$, let $(a, b) = \{x \in \boldsymbol{R} : a < x < b\}$ and $[a, b] = \{x \in \boldsymbol{R} : a \leq x \leq b\}$. Let $f : (-1, 1) \to \boldsymbol{R}$ ($x \mapsto \frac{x}{1-x^2}$), i.e., $f(x) = x/(1 - x^2)$ on the domain $(-1, 1)$. Show the following. (5 pts x 4 = 20 pts)

   (a) The function f is one-to-one.

   (b) The function f is onto.

   (c) An open interval $(-1, 1)$ and a closed interval $[-1, 1]$ are numerically equivalent.

   (d) For any $a, b \in \boldsymbol{R}$ with $a < b$, a closed interval $[a, b]$ and $\boldsymbol{R}$ are numerically equivalent.

ID#: Name:

6. Let $X = \boldsymbol{N} \times \boldsymbol{N}$ and $R = \{((a,b),(c,d)) \mid (a,b),(c,d) \in X,\ (a,b) \sim (c,d)\}$, where $(a,b) \sim (c,d) \Leftrightarrow ad = bc$.

   (a) State the definition of equivalence relation on a set A. (5 pts)

   (b) Show that R is an equivalence relation on X. (10 pts)

   (c) Let $Y = \{[(a,b)] \mid (a,b) \in X\}$ be the set of all distinct equivalence classes, where $[(a,b)]$ denotes the equivalence class containing (a,b), and let $\boldsymbol{Q}^+$ be the set of positive rational numbers. Then $f : Y \to \boldsymbol{Q}^+ ([(a,b)] \mapsto a/b)$ is a bijection. (5 pts)

Please write your comments:

(1) About this course, especially suggestions for improvements.

(2) Topics in Mathematics or in other subjects you want to study.

BCM I: Solutions to Final 2018

June 20, 2018

1. Let P, Q, R be statements. (5 pts x 2 = 10 pts)

   (a) Complete the following truth table.

P	Q	R	$(P$	$\vee$	$\sim Q)$	$\Rightarrow$	R	$(\sim P$	$\vee$	$R)$	$\wedge$	$(Q$	$\vee$	$R)$
T	T	T				$\boldsymbol{T}$					$\boldsymbol{T}$			
T	T	F				$\boldsymbol{F}$			F		$\boldsymbol{F}$			
T	F	T				$\boldsymbol{T}$					$\boldsymbol{T}$			
T	F	F				$\boldsymbol{F}$			F		$\boldsymbol{F}$		F	
F	T	T		F		$\boldsymbol{T}$					$\boldsymbol{T}$			
F	T	F		F		$\boldsymbol{T}$					$\boldsymbol{T}$			
F	F	T				$\boldsymbol{T}$					$\boldsymbol{T}$			
F	F	F				$\boldsymbol{F}$					$\boldsymbol{F}$		F	

   (b) Show $(P \vee \sim Q) \Rightarrow R \equiv (\sim P \vee R) \wedge (Q \vee R)$ by using formulas.

$$(P \vee \sim Q) \Rightarrow R \equiv \sim (P \vee \sim Q) \vee R \equiv (\sim P \wedge Q) \vee R \equiv (\sim P \vee R) \wedge (Q \vee R).$$

2. Show that there is an integer m such that for each integer $n \geq m$, there are positive integers a and b such that $n = 4a + 5b$. (10 pts)

   Soln. We apply 'Strong Form of Mathematical Induction'. Set $m = 21$. For $n = 21, 22, 23, 24$, we have

$$21 = 4 \cdot 4 + 5,\ 22 = 4 \cdot 3 + 5 \cdot 2,\ 23 = 4 \cdot 2 + 5 \cdot 3,\ 24 = 4 + 5 \cdot 4.$$

   Hence assume that $n \geq 25$. Then $n - 4 \geq 21 = m$, by induction hypothesis, there exist positive integers a' and b' such that $n - 4 = 4a' + 5b'$. Hence $n = 4(a' + 1) + 5b'$. Since $a = a' + 1$ and $b = b'$ are positive integers, $n = 4a + 5b$, as desired. ■

3. Let p be a prime number, let x, y and z be integers such that $x^2 + y^2 = pz^2$. Prove or disprove each of the following statements. (5 pts x 4 = 20 pts)

   (a) If p divides both x and y, then p divides z.

   Soln. [True] Suppose both x and y are divisible by p, then $x^2 + y^2 = pz^2$ is divisible by p^2. Hence z^2 is divisible by p. Since p is a prime number, p divides z. ■

   (b) If p does not divide y, there exists an integer w such that $w^2 + 1 \equiv 0 \pmod{p}$.

   Soln. [True] If y is not divisible by p, $\gcd(y, p) = 1$ and there are integers u, v such that $uy + vp = 1$. In particular, $uy \equiv 1 \pmod{p}$. Let $w = ux$. Then

$$w^2 + 1 \equiv (ux)^2 + 1 \equiv u^2x^2 + (uy)^2 \equiv u^2(x^2 + y^2) \equiv u^2 \cdot 0 \equiv 0 \pmod{p}.$$

   This proves the assertion. ■

   (c) If $p = 5$, i.e., $x^2 + y^2 = 5z^2$, then $x = y = z = 0$.

   Soln. [False] Let $x = 1$, $y = 2$ and $z = 1$. Then $x^2 + y^2 = 1^2 + 2^2 = 5 = 5z^2$. ■

(d) If $p = 7$, i.e., $x^2 + y^2 = 7z^2$, then $x = y = z = 0$.

Soln. [True] Consider the general case. Suppose not. Let $(x, y, z) \neq (0, 0, 0)$ be the solution such that $|z|$ is the smallest. If $z = 0$, then $x^2 + y^2 = 0$ and $x = y = z = 0$. Hence $|z| \neq 0$. By (a), either x or y is not divisible by p, as otherwise all of x, y, z are divisible by p and $(x/p, y/p, z/p) \neq (0, 0, 0)$ satisfies $(x/p)^2 + (y/p)^2 = p(z/p)^2$ and that $|z/p| < |z|$, a contradiction. By symmetry, we may assume that y is not divisible by p. Then by (b), there exists an integer w such that $w^2 + 1 \equiv 0 \pmod p$ or $w^2 \equiv -1 \pmod p$. Suppose $p = 7$, then $1^2 \equiv (-1)^2 \equiv 1 \pmod 7$, $2^2 \equiv (-2)^2 \equiv 4 \pmod 7$ and $3^2 \equiv (-3)^2 \equiv 2 \pmod 7$, and there is no w satisfying $w^2 \equiv -1 \equiv 6 \pmod 7$. ■

4. Let $f : X \to Y$, $g : Y \to Z$ and $h = g \circ f : X \to Z$ ($x \mapsto g(f(x))$) be functions. Prove or disprove the following. (5 pts x 4 = 20 pts)

(a) If f is onto and g is onto, then h is onto.

Soln. [True] Since $h : X \to Z$ with $h(x) = g(f(x))$, let $z \in Z$. We show that there exists $x \in X$ such that $h(x) = z$. Since $g : Y \to Z$ is onto and $z \in Z$, there exists $y \in Y$ such that $g(y) = z$. Since $y \in Y$ and $f : X \to Y$ is onto, there exists $x \in X$ such that $f(x) = y$. Hence $h(x) = g(f(x)) = g(y) = z$. ■

(b) If h is one-to-one, then g is one-to-one.

Soln. [False] $X = Z = \{1\}$ and $Y = \{1, 2\}$, $f : X \to Y$ is defined by $f(1) = 1$ and $g : Y \to Z$ is defined by $g(1) = g(2) = 1$. Then $h : X \to Z$ satisfies $h(1) = 1$ and it is one-to-one. However, g is not one-to-one as $g(1) = g(2)$. ■

(c) If h is one-to-one, then f is one-to-one.

Soln. [True] Since $f : X \to Y$, suppose $f(x) = f(x')$ with $x, x' \in X$. Then $h(x) = g(f(x)) = g(f(x')) = h(x')$. Since h is one-to-one by assumption, $x = x'$ and f is one-to-one. ■

(d) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ for all subsets A, B in Y.

Soln. [True] The following are all biconditional and $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ for all subsets A, B in Y.

$$x \in f^{-1}(A \cup B) \Leftrightarrow f(x) \in A \cup B \Leftrightarrow f(x) \in A \text{ or } f(x) \in B \Leftrightarrow x \in f^{-1}(A) \cup f^{-1}(B).$$

Therefore, $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$. ■

5. For $a, b \in \boldsymbol{R}$ with $a < b$, let $(a, b) = \{x \in \boldsymbol{R} : a < x < b\}$ and $[a, b] = \{x \in \boldsymbol{R} : a \leq x \leq b\}$. Let $f : (-1, 1) \to \boldsymbol{R}$ ($x \mapsto \frac{x}{1-x^2}$), i.e., $f(x) = x/(1 - x^2)$ on the domain $(-1, 1)$. Show the following. (5 pts x 4 = 20 pts)

(a) The function f is one-to-one.

Soln.

$$f'(x) = \frac{(1 - x^2) - x(-2x)}{(1 - x^2)^2} = \frac{1 + x^2}{(1 - x^2)^2} > 0 \quad \text{for all } x \in (-1, 1).$$

Hence $f(x)$ is strictly increasing and $f(x)$ is one-to-one. ■

You can show it without using Calculus. If $x > y$, then

$$f(x) - f(y) = \frac{x}{1 - x^2} - \frac{y}{1 - y^2} = \frac{x(1 - y^2) - y(1 - x^2)}{(1 - x^2)(1 - y^2)} = \frac{(x - y)(1 + xy)}{(1 - x^2)(1 - y^2)} > 0$$

as $-1 < y < x < 1$ and $|xy| < 1$. ■

(b) The function f is onto.

Soln. Since the domain is $(-1,1)$, f is continuous and

$$\lim_{x\to -1+0} f(x) = -\infty, \quad \lim_{x\to 1-0} f(x) = \infty,$$

By the Intermediate Value Theorem, for every $y \in \boldsymbol{R}$, there exists $x \in (-1,1)$ such that $f(x) = y$. ■

(c) An open interval $(-1,1)$ and a closed interval $[-1,1]$ are numerically equivalent.

Soln. Let $h : [-1,1] \to (-1,1)$ $(x \mapsto x/2)$ is one-to-one. Since $i : (-1,1) \to [-1,1]$ $(x \mapsto x)$ is one-to-one, by the Schröder-Bernstein Theorem, there is a bijection between $(-1,1)$ and $[-1,1]$ and these sets are numerically equivalent. ■

(d) For any $a, b \in \boldsymbol{R}$ with $a < b$, a closed interval $[a,b]$ and $\boldsymbol{R}$ are numerically equivalent.

Soln. Let $g : [a,b] \to [-1,1]$ $\left(x \mapsto -\dfrac{x-b}{a-b} + \dfrac{x-a}{b-a}\right)$. Then g is a bijection from $[a,b]$ to $[-1,1]$. Hence $[a,b]$ and $[-1,1]$ are numerically equivalent, $[-1,1]$ and $(-1,1)$ are numerically equivalent by (c) and $(-1,1)$ and $\boldsymbol{R}$ are numerically equivalent by (a) and (b). Since numerical equivalence is an equivalence relation, it is transitive, and $[a,b]$ and $\boldsymbol{R}$ are numerically equivalent. ■

6. Let $X = \boldsymbol{N} \times \boldsymbol{N}$ and $R = \{((a,b),(c,d)) \mid (a,b),(c,d) \in X,\ (a,b) \sim (c,d)\}$, where $(a,b) \sim (c,d) \Leftrightarrow ad = bc$.

(a) State the definition of equivalence relation on a set A. (5 pts)

Soln. An equivalence relation R on A is a subset of $A \times A$ satisfying the following three conditions. (i) $(a,a) \in R$ for all $a \in A$, (i) if $(a,b) \in R$, then $(b,a) \in R$, (iii) if (a,b) and (c,d) are in R, then (a,d) is in R. ■

(b) Show that R is an equivalence relation on X. (10 pts)

Soln. $X = \boldsymbol{N} \times \boldsymbol{N}$.

(i) For all $(a,b) \in X$, $ab = ba$ and $(a,b) \sim (a,b)$.

(ii) If $(a,b) \sim (c,d)$, then $ad = bc$. Hence $cb = da$ and $(c,d) \sim (a,b)$.

(iii) If $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$, then $ad = bc$ and $cf = de$. Hence $adcf = bcde$. Since all factors are in $\boldsymbol{N}$ and nonzero, $af = be$ and $(a,b) \sim (ef)$.

Therefore, R is an equivalence relation on X. ■

(c) Let $Y = \{[(a,b)] \mid (a,b) \in X\}$ be the set of all distinct equivalence classes, where $[(a,b)]$ denotes the equivalence class containing (a,b), and let $\boldsymbol{Q}^+$ be the set of positive rational numbers. Then $f : Y \to \boldsymbol{Q}^+([(a,b)] \mapsto a/b)$ is a bijection. (5 pts)

Soln. Note that $(a,b) \sim (c,d) \Leftrightarrow ad = bc \Leftrightarrow a/b = c/d \in \boldsymbol{Q}^+$. Hence (I) the function, $f : Y \to \boldsymbol{Q}^+([(a,b)] \mapsto a/b)$ is well-defined, i.e., if $[(a,b)] = [(c,d)]$, then $a/b = c/d$. (ii) It is one-to-one as $f([a,b]) = a/b = c/d = f([(c,d])$, then $[(a,b)] = [(c,d)]$ as $(a,b) \sim (c,d)$. (iii) It is onto, as every element in $\boldsymbol{Q}^+$ can be expressed as a/b with $(a,b) \in \boldsymbol{N} \times \boldsymbol{N}$. ■