Algebra II Review[1]

1. Let R be a commutative ring with idetity. Show the following. [Theorem 12.1]
   (a) $0a = 0$ for all $a \in R$.
   (b) $(-a)(-b) = ab$ for all $a, b \in R$.

2. Let $R = \boldsymbol{Z}_{18}$. (Try a more general case $R = \boldsymbol{Z}_n$.)
   (a) Find all zero divisors of R.
   (b) Find all units of R.
   (c) Show that every nonzero element of R is a unit or a zero-divisor. [Ex 13.5]
   (d) Prove or disprove that the unit group $U(R)$ is a cyclic group.
   (e) Find all idempotents $e \in R$ such that $e^2 = e$. [Ex 13.18]
   (f) Find all prime ideals and maximal ideals of R.
   (g) Every ideal I of R has the form $I = \langle a \rangle$ for some $a \in I$. [Ex. 14.41, 57]
   (h) Find all ring homomorphisms from R into R. [Ex.15.8]
   (i) Find all ring isomorphisms from R to itself. [Ex.15.22]

3. Let R be an integral domain and let $a, b \in R$. Show that the following are equivalent.
   (i) $a \mid b$ and $b \mid a$.
   (ii) $\langle a \rangle = \langle b \rangle$.
   (iii) $a \approx b$, i.e., there exists $u \in U(R)$ such that $b = ua$. where u is a unit of R.

4. Let R be a commutative ring with unity and let A be an ideal of R. Then
   (a) If $A \cap U(R) \neq \emptyset$, then $A = R$. [Ex. 14.17]
   (b) If R is a field, then R is an integral domain.
   (c) R/A is an integral domain if and only if A is prime. [Theorem 14.3]
   (d) R/A is a field if and only if A is maximal. [Theorem 14.4]
   (e) If A is maximal, A is prime.

5. Let R be an integral domain, $R[x]$ and $R[x, y]$ rings of polynomials over R.
   (a) Define $\deg f$ for $f \in R[x]$ and show that for $f, g \in R[x]$, $\deg f \cdot g = \deg f + \deg g$ holds including the case $f = 0$ or $g = 0$.
   (b) Show that $R[x, y]$ is an integral domain.
   (c) Show that $U(R[x, y]) = U(R)$. Here $U(R[x, y])$ and $U(R)$ denote the set of units (invertible elements) of $R[x, y]$ and R respectively.
   (d) Let $f(x, y), g(x, y) \in R[x, y]$. Show that if the ideals generated by $f(x, y)$ and $g(x, y)$ are equal, i.e., $R[x, y]f(x, y) = R[x, y]g(x, y)$, then there exists $a \in U(R)$ such that $f(x, y) = a \cdot g(x, y)$.

[1] For references, see 'Contemporary Abstract Algebra, Eighth Edition' by Joseph A. Gallian

(e) Show that $R[x,y]$ is not a principal ideal domain.[2]

(f) $A = \{f(x) \in R[x] \mid f(0) = 0\}$ is a prime ideal. [Example 14.17]

(g) A above is a maximal ideal if and only if R is a field. [Example 14.17, Ex 14.34, Ex 16.20]

6. Let R and R' be rings with an identity element. Let $f : R \longrightarrow R'$ be a ring homomorphism from R to R', i.e., f satisfies $f(a+b) = f(a)+f(b)$, $f(ab) = f(a)f(b)$, $(a, b \in R)$. In addition assume that $f(1) = 1$. Let J be a (two-sided) ideal of R'. Show the following.

   (a) $f(0) = 0$ and $f(-a) = -f(a)$. [Theorem 10.1]

   (b) $f^{-1}(J)$ is a two-sided ideal of R. [Theorem 15.1]

   (c) If J is a prime ideal, then $f^{-1}(J)$ is a prime ideal. [Ex 15.47]

   (d) If $f^{-1}(J) \cap U(R) \neq \emptyset$, then $J = R'$. [Ex.14.17]

7. Let R be a commutative ring with identity, and I, J, K be its ideals. Suppose $I + J = I + K = J + K = R$. Show the following.

   (a) $IJ = I \cap J$. Recall that IJ is the set of finite sums of the products of elements in I and J, i.e., $IJ = \{\sum_i x_i \cdot y_i \mid x_i \in I, \text{ and } y_i \in J\}$. [Ex 14.14, 16]

   (b) Let $\phi : R \to R/I \times R/J$ $(x \mapsto (x+I, x+J))$. Show that ϕ is surjective. [Suppl. Ex 6, p.347]

   (c) $IJK = I \cap J \cap K$.

8. Let R be an integral domain and p a non zero, non unit element of R.

   (a) Suppose that $\langle p \rangle$ is a prime ideal. Show that p is an irreducible element.[Theorem 18.1]

   (b) Suppose that R is a UFD and that p is an irreducible element. Show that $\langle p \rangle$ is a prime ideal. [Ex.18.39]

9. Let $\boldsymbol{Z}[x]$ be a polynomial ring in x over $\boldsymbol{Z}$. Let

$$\phi : \boldsymbol{Z}[x] \longrightarrow \boldsymbol{C} \ (f(x) \mapsto f(\sqrt{-13})).$$

   (You may assume that ϕ is a ring homomorphism.)

   (a) Let $R = \mathrm{Im}\phi$. Show that R is a subring of $\boldsymbol{C}$ and R is an integral domain.

   (b) Show that $R = \{a + b\sqrt{-13} \mid a, b \in \boldsymbol{Z}\}$.

   (c) For $\alpha = a + b\sqrt{-13} \in R$, let $N(\alpha) = N(a + b\sqrt{-13}) = a^2 + 13b^2$. Show that for $\alpha, \beta \in R$, $N(\alpha \cdot \beta) = N(\alpha)N(\beta)$.

   (d) Show that

$$\alpha \in U(R) \Leftrightarrow N(\alpha) = 1 \Leftrightarrow \alpha \in \{1, -1\}.$$

   (e) Show that 2 is an irreducible element of R.

   (f) Show that R is not a UFD.[3]

[2] Hint: Proposition 7.2 (Theorem 18.2)

[3] Show by definition of UFD or use Ex.18.39.

10. Let F be a field and $F[x]$ a polynomial ring over F. Let E be an extension field of F and $a \in E$ algebraic over F. Suppose $p(x)$ is a monic (i.e., the leading coefficient is 1) polynomial in $F[x]$ of minimal degree such that $p(x) = 0$. Let $\phi : F[x] \to E$ $(f(x) \mapsto f(a))$.[4]

    (a) Show that $p(x)$ is irreducible.

    (b) If $q(x) \in F[x]$ is a monic irreducible polynomial such that $q(a) = 0$, then $p(x) = q(x)$.

    (c) For $f(x) \in F[x]$, $f(a) = 0$ if and only if $p(x) \mid f(x)$.

    (d) For $f(x) \in F[x]$, if $f(a) \neq 0$, then there exists $g(x), h(x) \in F[x]$ such that $f(x)g(x) + h(x)p(x) = 1$.

    (e) $\mathrm{Ker}(\phi) = \langle p(x) \rangle$.

    (f) $\mathrm{Ker}(\phi)$ is a maximal ideal and $\phi(F[x]) = \{f(a) \mid f(x) \in F[x]\} = F(a)$ is the smallest subfield of E containing F and a.

    (g) Suppose $n = \deg p(x)$. Then $\phi(F[x]) = \{\alpha_0 + \alpha_1 a + \cdots + \alpha_{n-1} a^{n-1} \mid \alpha_0, \alpha_1, \ldots, \alpha_{n-1}\}$ and $1, a, a^2, \ldots, a^{n-1}$ are linearly independent over F.

11. Let $F = \boldsymbol{Q}(\sqrt{2} + \sqrt{3})$ be a subfield of $\boldsymbol{C}$.

    (a) Show that $F = \boldsymbol{Q}(\sqrt{2}, \sqrt{3})$.

    (b) Show that $[F : \boldsymbol{Q}] = 4$.

    (c) Show that $x^2 - 3$ is irreducible over $\boldsymbol{Q}(\sqrt{2})$.

    (d) Find the minimal polynomial of $\sqrt{2} + \sqrt{3}$ over $\boldsymbol{Q}$.

Recommended Exercises for Review

Ch.12. 23, 24, 25, 26

Ch.13. 7, 26, 49 (See 15.44)

Ch.14. 22, 34, 37

Suppl.12-14. 4, 19, 24, 29

Ch.15. 8, 22, 44, 47

Ch.16. 17, 19, 23, 24, 42

Ch.17. 6, 30, 31, 32

Ch.18. 1, 2, 13, 15, 22, 23

Suppl.15-18. 2, 6, 8, 22

Ch.19. 9, 10, 30

Ch.20. 1, 2, 27, 35

Ch.21. 1, 20, 35

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[4]First show (a)–(d) only by 'Division Algorithm for $F[x]$. Next show (e)–(g) by using (a)–(d). Finally prove (e)–(g) first by results developed by ring theory and show (a)– (d) as well.