Algebra I: Final 2010

June 21, 2010

Division: **ID#:** **Name:**

Quote the following when necessary.

Subgroup H of a group G:

$$H \leq G \Leftrightarrow \emptyset \neq H \subseteq G,\ xy \in H \ \text{ and } x^{-1} \in H \ \text{ for all } x, y \in H.$$

Order of an Element: Let g be an element of a group G. Then $\langle g \rangle = \{g^n \mid n \in \boldsymbol{Z}\}$ is a subgroup of G. If there is a positive integer m such that $g^m = e$, where e is the identity element of G, $|g| = \min\{m \mid g^m = e,\ m \in \boldsymbol{N}\}$ and $|g| = |\langle g \rangle|$. Moreover, for any integer n, $|g|$ divides n if and only if $g^n = e$.

G/H and $|G : H|$: If $H \leq G$, $G/H = \{gH \mid g \in G\}$, i.e., the set of left cosets of H in G and $|G/H|$ is denoted by $|G : H|$.

Lagrange's Theorem: If H is a subgroup of a finite group G, then $|G| = |G : H||H|$.

Normal Subgroup: A subgroup H of a group G is normal if $gHg^{-1} = H$ for all $g \in G$. If H is a normal subgroup of G, then G/H becomes a group with respect to the binary operation $(gH)(g'H) = gg'H$.

Center of a Group: The center $Z(G)$ of a group G is the set $\{x \in G \mid gxg^{-1} = x \text{ for all } g \in G\}$.

Direct Product: If $\gcd\{m, n\} = 1$, then $\boldsymbol{Z}_{mn} \approx \boldsymbol{Z}_m \oplus \boldsymbol{Z}_n$ and $U(mn) \approx U(m) \oplus U(n)$.

Kernel: If $\phi : G \to \overline{G}$ is a group homomorphism, $\mathrm{Ker}(\phi) = \{x \in G \mid \phi(x) = e_{\overline{G}}\}$, where $e_{\overline{G}}$ is the identity element of $\overline{G}$.

1. Let H be a subgroup of a group G. Show that for $x, y \in G$, $xH = yH \Leftrightarrow x^{-1}y \in H$. (10 pts)

Division: **ID#:** **Name:**

2. Let H and K be subgroups of a group G. Show the following. (25 pts)

   (a) $H \cap K \leq G$.

   (b) $\phi : H/(H \cap K) \mapsto G/K$ $(h(H \cap K) \mapsto hK)$ is a well-defined injection (i.e., one-to-one mapping).

   (c) If $|H|$ and $|K|$ are finite, $|HK||H \cap K| = |H||K|$.

Division: **ID#:** **Name:**

3. Let $\phi : G \to \overline{G}$ is a group homomorphism from G to $\overline{G}$. Show the following. (25 pts)

   (a) $\phi(e_G) = e_{\overline{G}}$ and $\phi(x^{-1}) = \phi(x)^{-1}$ for all $x \in G$, where e_G is the identity element of G and $e_{\overline{G}}$ is that of $\overline{G}$.

   (b) If $H \leq G$, then $\phi(H) \leq \overline{G}$.

   (c) If the order of an element g is finite, then $|\phi(g)| \mid |g|$.

   (d) Let $K = \text{Ker}(\phi)$. If $S \subset G$, then $\phi^{-1}(\phi(S)) = SK$.

Division: **ID#:** **Name:**

4. Answer the following questions on Abelian groups of order $108 = 2^2 \cdot 3^3$. (25 pts)

   (a) Using the Fundamental Theorem of Finite Abelian Groups and list all non-isomorphic Abelian groups of order 108 and give a brief explanation.

   (b) List all Abelian groups of order 108 in your list above that have a group homomorphism onto $\boldsymbol{Z}_9$. Give your reason.

   (c) Determine whether or not $U(133) \approx U(324)$. Give your reason.

Division: **ID#:** **Name:**

5. The following is the Cayley table of D_4. (15 pts)

	R_0	$R_{\pi/2}$	R_π	$R_{3\pi/2}$	H	V	D	D'
R_0	R_0	$R_{\pi/2}$	R_π	$R_{3\pi/2}$	H	V	D	D'
$R_{\pi/2}$	$R_{\pi/2}$	R_π	$R_{3\pi/2}$	R_0	D'	D	H	V
R_π	R_π	$R_{3\pi/2}$	R_0	$R_{\pi/2}$	V	H	D'	D
$R_{3\pi/2}$	$R_{3\pi/2}$	R_0	$R_{\pi/2}$	R_π	D	D'	V	H
H	H	D	V	D'	R_0	R_π	$R_{\pi/2}$	$R_{3\pi/2}$
V	V	D'	H	D	R_π	R_0	$R_{3\pi/2}$	$R_{\pi/2}$
D	D	V	D'	H	$R_{3\pi/2}$	$R_{\pi/2}$	R_0	R_π
D'	D'	H	D	V	$R_{\pi/2}$	$R_{3\pi/2}$	R_π	R_0

As for the following no explanation is required.

(a) Find $Z(D_4)$.

(b) Find all normal subgroups different from $\{R_0\}, Z(D_4)$ and D_4.

(c) Find all subgroups of D_4 that are not normal.

Please write your message: Comments on group theory. Suggestions for improvements of this course. Write on the back of this sheet is also welcome.

Algebra I: Final 2010 Solutions

June 21, 2010

1. Let H be a subgroup of a group G. Show that for $x, y \in G$, $xH = yH \Leftrightarrow x^{-1}y \in H$. (10 pts)

   Solution. Since $H \leq G$, $H \neq \emptyset$. Let $x \in H$. Then $x^{-1} \in H$ and $e = xx^{-1} \in H$.

   Suppose $xH = yH$. Since $e \in H$, $xH = yH$ implies that $y = ye \in yH = xH$. Hence there exists $h \in H$ such that $y = xh$. Therefore by multiplying x^{-1} to both hand sides from left, $x^{-1}y = h \in H$.

   Conversely let $x^{-1}y = h \in H$. Then $y = xh$ and

   $$yH = xhH \subset xH = xeH = xx^{-1}yh^{-1}H \subset yH.$$

   Therefore $xH = yH$. ■

2. Let H and K be subgroups of a group G. Show the following. (25 pts)

   (a) $H \cap K \leq G$.

   (b) $\phi : H/(H \cap K) \mapsto G/K \ (h(H \cap K) \mapsto hK)$ is a well-defined injection (i.e., one-to-one mapping).

   (c) If $|H|$ and $|K|$ are finite, $|HK||H \cap K| = |H||K|$.

   Solution.

   (a) By 1, $e \in H$ and $e \in K$. Hence $H \cap K \neq \emptyset$. Suppose $x, y \in H \cap K$. Since $x, y \in H$ and H is a subgroup of G, $xy \in H$ and $x^{-1} \in H$. Similarly since K is a subgroup of G, $xy \in K$ and $x^{-1} \in K$. Therefore $xy \in H \cap K$ and $x^{-1} \in H \cap K$. Thus $H \cap K$ satisfies the condition of a subgroup of G. ■

   (b) Note that by the definition of subgroups, $H \cap K \leq H$. Hence we can apply 1 and for all $h, h' \in H$ we have

   $$h(H \cap K) = h'(H \cap K) \Leftrightarrow h^{-1}h' \in H \cap K \Leftrightarrow hK = h'K.$$

   Since $h(H \cap K) = h'(H \cap K)$ implies $hK = h'K$, ϕ is well-defined mapping from $H/(H \cap K)$ to G/K. Moreover since $hK = h'K$ implies $h(H \cap K) = h'(H \cap K)$, $hK = \phi(h(H \cap K)) = \phi(h'(H \cap K)) = h'K$ implies $h(H \cap K) = h'(H \cap K)$ and ϕ is an injection. ■

   (c) For each $h \in H$, we claim that $\psi : K \to hK \ (k \mapsto hk)$ is a bijection. This is an injection because $hk = \psi(k) = \phi(k') = hk'$ implies $k = k'$ for all $k, k' \in K$ by multiplying h^{-1} to both hand sides from the left. This is a surjection because for $k \in K$, $\psi(k) = hk$. This proves our claim and $|hK| = |K|$ for all $h \in H$.

   Since ϕ is an injection, there are $|H : H \cap K|$ cosets of K in G in the image of ϕ, that is HK contains $|H : H \cap K|$ cosets of K in G. By our claim above, each coset hK contains $|K|$ elements. Therefore by Lagrange's Theorem,

   $$|HK| = |H : H \cap K||K| = \frac{|H|}{|H \cap K|}|K|, \text{ and hence } |HK||H \cap K| = |H||K|. \quad ■$$

3. Let $\phi : G \to \overline{G}$ is a group homomorphism from G to $\overline{G}$. Show the following. (25 pts)

   (a) $\phi(e_G) = e_{\overline{G}}$ and $\phi(x^{-1}) = \phi(x)^{-1}$ for all $x \in G$, where e_G is the identity element of G and $e_{\overline{G}}$ is that of $\overline{G}$.

   (b) If $H \leq G$, then $\phi(H) \leq \overline{G}$.

   (c) If the order of an element g is finite, then $|\phi(g)| \mid |g|$.

   (d) Let $K = \mathrm{Ker}(\phi)$. If $S \subset G$, then $\phi^{-1}(\phi(S)) = SK$.

Solution.

   (a) $e_{\overline{G}} = \phi(e_G)^{-1}\phi(e_G) = \phi(e_G)^{-1}\phi(e_G e_G) = \phi(e_G)^{-1}\phi(e_G)\phi(e_G) = \phi(e_G)$. Moreover
   $\phi(x^{-1}) = \phi(x^{-1})\phi(x)\phi(x)^{-1} = \phi(x^{-1}x)\phi(x)^{-1} = \phi(e_G)\phi(x)^{-1} = e_{\overline{G}}\phi(x)^{-1} = \phi(x)^{-1}$.
   This proves both assertions. ■

   (b) Since $e_G \in H$ by the solution of 1, $e_{\overline{G}} = \phi(e_G) \in \phi(H)$ and $\phi(H) \neq \emptyset$. Let $x, y \in \phi(H)$. Then there exist $x', y' \in H$ such that $x = \phi(x')$ and $y = \phi(y')$. Since ϕ is a group homomorphism $xy = \phi(x')\phi(y') = \phi(x'y') \in \phi(H)$ as H is a subgroup of G. By (a) $x^{-1} = \phi(x')^{-1} = \phi(x'^{-1}) \in \phi(H)$ as H is a subgroup of G. Therefore $\phi(H)$ satisfies the condition of a subgroup of $\overline{G}$ and $\phi(H) \leq \overline{G}$. ■

   (c) Let $n = |g|$. Then $n = |\langle g \rangle|$. $g^n = e_G$. Hence by (a) $e_{\overline{G}} = \phi(e_G) = \phi(g^n) = \phi(g)^n$. Therefore $|\phi(g)|$ divides $n = |g|$. ■

   (d) Let $s \in S$ and $k \in K = \mathrm{Ker}(\phi)$. Then $\phi(sk) = \phi(s)\phi(k) = \phi(s)e_{\overline{G}} = \phi(s) \in \phi(S)$. Hence $sk \in \phi^{-1}(\phi(S))$ and $SK \subset \phi^{-1}(\phi(S))$. Conversely if $x \in \phi^{-1}(\phi(S))$, $\phi(x) \in \phi(S)$. Hence there exists $s \in S$ such that $\phi(x) = \phi(s)$. Therefore $e_{\overline{G}} = \phi(s)^{-1}\phi(x) = \phi(s^{-1}x)$ and $s^{-1}x \in K$. Let $s^{-1}x = k$ for some $k \in K$. Then $x = sk \in SK$. This proves $\phi^{-1}(\phi(S)) \subset SK$ and $\phi^{-1}(\phi(S)) = SK$. ■

4. Answer the following questions on Abelian groups of order $108 = 2^2 \cdot 3^3$. (25 pts)

   (a) Using the Fundamental Theorem of Finite Abelian Groups and list all non-isomorphic Abelian groups of order 108 and give a brief explanation.

   (b) List all Abelian groups of order 108 in your list above that have a group homomorphism onto $\boldsymbol{Z}_9$. Give your reason.

   (c) Determine whether or not $U(133) \approx U(324)$. Give your reason.

Solution.

   (a) Each group of order 108 is an internal direct sum of groups of order 2^2 and 3^3 and each group of order 2^2 and 3^3 are isomorphic to an external direct product of cyclic groups of decreasing order and if the sequence of orders of such cyclic groups are distinct then they are non-isomorphic. Hence they are

$$\begin{array}{ccc}
\boldsymbol{Z}_4 \oplus \boldsymbol{Z}_{27} & \approx & \boldsymbol{Z}_{108} \\
\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_{27} & \approx & \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_{54} \\
\boldsymbol{Z}_4 \oplus \boldsymbol{Z}_9 \oplus \boldsymbol{Z}_3 & \approx & \boldsymbol{Z}_3 \oplus \boldsymbol{Z}_{36} \\
\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_9 \oplus \boldsymbol{Z}_3 & \approx & \boldsymbol{Z}_6 \oplus \boldsymbol{Z}_{18} \\
\boldsymbol{Z}_4 \oplus \boldsymbol{Z}_3 \oplus \boldsymbol{Z}_3 \oplus \boldsymbol{Z}_3 & \approx & \boldsymbol{Z}_3 \oplus \boldsymbol{Z}_3 \oplus \boldsymbol{Z}_{12} \\
\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_3 \oplus \boldsymbol{Z}_3 \oplus \boldsymbol{Z}_3 & \approx & \boldsymbol{Z}_3 \oplus \boldsymbol{Z}_6 \oplus \boldsymbol{Z}_6
\end{array}$$

(b) By 3(c), the group must have an element of order 9. Hence they are

$$\boldsymbol{Z}_{108},\ \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_{54},\ \boldsymbol{Z}_3 \oplus \boldsymbol{Z}_{36},\ \text{and}\ \boldsymbol{Z}_6 \oplus \boldsymbol{Z}_{18}.$$

(c) $U(133) = U(7 \cdot 19) \approx U(7) \oplus U(19) \approx \boldsymbol{Z}_6 \oplus \boldsymbol{Z}_{18}$, while $U(324) = U(4 \cdot 81) \approx U(4) \oplus U(81) \approx \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_{54}$. Hence these groups of order 108 are not isomorphic. (Since it is easy to see that the first does not have an element of order 9 by the first direct sum decomposition, the only thing to show is that $U(81)$ has an element of order at least 7 and 2 is such an element. ■

5. The following is the Cayley table of D_4. (15 pts)

	R_0	$R_{\pi/2}$	R_π	$R_{3\pi/2}$	H	V	D	D'
R_0	R_0	$R_{\pi/2}$	R_π	$R_{3\pi/2}$	H	V	D	D'
$R_{\pi/2}$	$R_{\pi/2}$	R_π	$R_{3\pi/}$	R_0	D'	D	H	V
R_π	R_π	$R_{3\pi/2}$	R_0	$R_{\pi/2}$	V	H	D'	D
$R_{3\pi/2}$	$R_{3\pi/2}$	R_0	$R_{\pi/2}$	R_π	D	D'	V	H
H	H	D	V	D'	R_0	R_π	$R_{\pi/2}$	$R_{3\pi/2}$
V	V	D'	H	D	R_π	R_0	$R_{3\pi/2}$	$R_{\pi/2}$
D	D	V	D'	H	$R_{3\pi/2}$	$R_{\pi/2}$	R_0	R_π
D'	D'	H	D	V	$R_{\pi/2}$	$R_{3\pi/2}$	R_π	R_0

As for the following no explanation is required.

(a) Find $Z(D_4)$.

(b) Find all normal subgroups different from $\{R_0\}, Z(D_4)$ and D_4.

(c) Find all subgroups of D_4 that are not normal.

Solution.

(a) $Z(D_4) = \{R_0, R_\pi\}$.

(b) If its order is 2, it is in the center. Hence the only possibility is of order 4. Since $D_4/Z(D_4) \approx \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2$, there are three such groups. They are

$$\{R_0, R_{\pi/2}, R_\pi, R_{3\pi/2}\}, \{R_0, R_\pi, H, V\}, \{R_0, R_\pi, D, D'\}.$$

(c) Since all subgroups of order $1, 4, 8$ are normal. The only possibility is the one of order 2 which is not in the center. Hence

$$\{R_0, H\}, \{R_0, V\}, \{R_0, D\}, \{R_0, D'\}.$$