Algebra I: Final 2008

June 23, 2008

Division: **ID#:** **Name:**

1. Let H be a subgroup of a finite group G, i.e., $\emptyset \neq H \subseteq G$ and

$$xy \in H \text{ and } x^{-1} \in H \text{ for all } x, y \in H.$$

(a) Show that $gHg^{-1} \leq G$ for every $g \in G$.

(b) Show that $|H| = |gHg^{-1}|$ for all $g \in G$.

(c) Suppose that $N \lhd G$. Show that $HN \leq G$.

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2. Let G be a finite group and X the set of all subgroups of G. Let

$$\alpha : G \times X \to X \ ((g, H) \mapsto gHg^{-1}).$$

(a) Show that α defines a left action of G on the set X.

(b) Let $H \leq G$ and $N = N_G(H) = \{g \in G \mid gHg^{-1} = H\}$. Show that $N \leq G$, and $xHx^{-1} = yHy^{-1}$ if and only if $xN = yN$.

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3. Let H and K be subgroups of a finite group G. Let $\alpha : H \times K \to G$ $((h,k) \mapsto hk)$.

   (a) Show that α is a homomorphism if and only if $hk = kh$ for all $h \in H$, $k \in K$.

   (b) Suppose $H \lhd G$, $K \lhd G$ and $\gcd(|H|, |K|) = 1$, i.e., both H and K are normal subgroups of G and $|H|$ and $|K|$ are coprime. Show that α is an injective homomorphism.

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Sylow's Theorem

Let G be a finite group and let p^a denote the largest power of the prime p dividing $|G|$. Let $\mathrm{Syl}_p(G)$ denote the set of all Sylow p-subgroups of G, i.e., subgroups of order p^a, and $P \in \mathrm{Syl}_p(G)$. Then the following hold.

(i) $|\mathrm{Syl}_p(G)| = |G : N_G(P)| \equiv 1 \pmod p$.

(ii) Let Q be a p-subgroup of G and P a Sylow p-subgroup of G, then there is $g \in G$ such that $Q \subseteq gPg^{-1}$. In particular, any two Sylow p-subgroups are conjugate in G.

Groups of Order p^2

Let p be a prime number, and G a group of order p^2. Then G is abelian, i.e., commutative.

4. Let G be a group of order $90 = 2 \cdot 3^2 \cdot 5$. Let P be a Sylow 5-subgroup of G, i.e., a subgroup of G of order 5.

   (a) Show that if $P \ntrianglelefteq G$ then there is a subgroup of G of order 15.

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(b) Show that G always has a cyclic subgroup H of order 15.

(c) Show that G is not simple, i.e., G has a nontrivial normal subgroup. (Hint: Let R be a subgroup of order 3 in H, and consider $N_G(R)$.)

Please write your message: Comments on group theory. Suggestions for improvements of this course.

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1. Let H be a subgroup of a finite group G, i.e., $\emptyset \neq H \subseteq G$ and

$$xy \in H \text{ and } x^{-1} \in H \text{ for all } x, y \in H.$$

(a) Show that $gHg^{-1} \leq G$ for every $g \in G$.
Solution. Since H is a subgroup of G, $1 \in H$. Hence $1 = g1g^{-1} \in gHg^{-1}$. Let $x, y \in gHg^{-1}$. Then there exist $h, h' \in H$ such that $x = ghg^{-1}$ and $y = gh'g^{-1}$. Since $H \leq G$, $hh' \in H$ and $h^{-1} \in H$. Hence

$$xy = ghg^{-1}gh'g^{-1} = ghh'g^{-1} \in gHg^{-1}, \text{ and } x^{-1} = (ghg^{-1})^{-1} = gh^{-1}g^{-1} \in gHg^{-1}.$$

Therefore $gHg^{-1} \leq G$. ■

(b) Show that $|H| = |gHg^{-1}|$ for all $g \in G$.
Solution. Let $\alpha : H \to gHg^{-1}$ $(h \mapsto ghg^{-1})$. We show that α is a bijection. If $\alpha(h) = \alpha(h')$ for $h, h' \in H$, then $ghg^{-1} = \alpha(h) = \alpha(h') = gh'g^{-1}$. By multiplying g^{-1} from the left and g from the right we have $h = g^{-1}ghg^{-1}g = g^{-1}gh'g^{-1}g = h'$. Hence α is injective. Let $x \in gHg^{-1}$. Then there exists $h \in H$ such that $x = ghg^{-1}$ and $\alpha(h) = ghg^{-1} = x$. Hence α is surjective. Therefore α is a bijection from H to gHg^{-1} and $|H| = |gHg^{-1}|$. ■

(c) Suppose that $N \lhd G$. Show that $HN \leq G$.
Solution. Since H and N are subgroups of G, $1 = 1 \cdot 1 \in HN$. Let $x, y \in HN$. Then there exist $h, h' \in H$ and $n, n' \in N$ such that $x = hn$ and $y = h'n'$. Hence

$$xy = hnh'n' = hh'h'^{-1}nh'n' \in HHh'^{-1}Nh'N \subseteq HNN \subseteq HN,$$

and

$$x^{-1} = (hn)^{-1} = n^{-1}h^{-1} = h^{-1}hn^{-1}h^{-1} \in HhNh^{-1} = HN.$$

Therefore $HN \leq G$. ■

2. Let G be a finite group and X the set of all subgroups of G. Let

$$\alpha : G \times X \to X \ ((g, H) \mapsto gHg^{-1}).$$

(a) Show that α defines a left action of G on the set X.
Solution. Let $g \cdot H := gHg^{-1}$. For any subgroup H of G, $gHg^{-1} \leq G$ by 1(a). Hence $g \cdot H \in X$ for all $g \in G$ and $H \in X$. We show that for $g_1, g_2 \in G$, $g_1 \cdot (g_2 \cdot H) = (g_1g_2) \cdot H$ and $1 \cdot H = H$. Let $g_1, g_2 \in G$. Then

$$g_1 \cdot (g_2 \cdot H) = g_1 \cdot (g_2Hg_2^{-1}) = g_1(g_2Hg_2^{-1})g_1^{-1} = (g_1g_2)H(g_1g_2)^{-1} = (g_1g_2) \cdot H,$$

and $1 \cdot H = 1H1^{-1} = H$. Therefore α defines a left action of G on the set X. ■

(b) Let $H \leq G$ and $N = N_G(H) = \{g \in G \mid gHg^{-1} = H\}$. Show that $N \leq G$, and $xHx^{-1} = yHy^{-1}$ if and only if $xN = yN$.

Solution. Since $1H1^{-1} = H$, $1 \in N$ by definition. Let $g_1, g_2 \in N$. Then $g_1Hg_1^{-1} = H$ and $g_2Hg_2^{-1} = H$ by definition. Hence $g_1g_2H(g_1g_2)^{-1} = g_1(g_2Hg_2^{-1})g_1^{-1} = g_1Hg_1^{-1} = H$, and $g_1^{-1}H(g_1^{-1})^{-1} = g_1^{-1}Hg = g_1^{-1}g_1Hg_1^{-1}g = H$. Therefore $g_1g_2 \in N$, $g_1^{-1} \in N$, and hence $N \leq G$.

Since $N \leq G$, note that $xN = yN$ if and only if $x^{-1}y \in N$. (If $xN = yN$, then $x^{-1}y = x^{-1}y1 \in x^{-1}yN = x^{-1}xN = N$. If $x^{-1}y \in N$, then $xN = xx^{-1}y(x^{-1}y)^{-1}N \subseteq yNN \subseteq yN = yy^{-1}xx^{-1}yN \subseteq xNN \subseteq xN$. Hence $xN = yN$.)

Suppose $xHx^{-1} = yHy^{-1}$. Then $H = x^{-1}yHy^{-1}x = x^{-1}yH(x^{-1}y)^{-1}$. Hence $x^{-1}y \in N$. Thus by our note above, $xN = yN$.

Conversely suppose $xN = yN$. Then again by our note, $x^{-1}y \in N$. Therefore $H = x^{-1}yH(x^{-1}y)^{-1} = x^{-1}yHy^{-1}x$. By multiplying x from the left and x^{-1} from the right, we have $xHx^{-1} = yHy^{-1}$. ■

3. Let H and K be subgroups of a finite group G. Let $\alpha : H \times K \to G$ $((h,k) \mapsto hk)$.

(a) Show that α is a homomorphism if and only if $hk = kh$ for all $h \in H$, $k \in K$.

Solution. Suppose $hk = kh$ for all $h \in H$ and $k \in K$. Let $h_1, h_2 \in H$ and $k_1, k_2 \in K$. Then by our assumption, $h_2k_1 = k_1h_2$. Hence

$$\alpha((h_1,k_1)(h_2,k_2)) = \alpha(h_1h_2, k_1k_2) = h_1h_2k_1k_2 = h_1k_1h_2k_2 = \alpha(h_1,k_1)\alpha(h_2,k_2).$$

Conversely assume that α is a homomorphism and let $h \in H$ and $k \in K$. Then

$$hk = \alpha(h,k) = \alpha((1,k)(h,1)) = \alpha(1,k)\alpha(h,1) = kh.$$

This proves the assertions.

(b) Suppose $H \lhd G$, $K \lhd G$ and $\gcd(|H|,|K|) = 1$, i.e., both H and K are normal subgroups of G and $|H|$ and $|K|$ are coprime. Show that α is an injective homomorphism.

Solution. Since both H and K are subgroups of G, so is $H \cap K$. By a theorem of Lagrange, $|H \cap K|$ divides both $|H|$ and $|K|$. Since $\gcd(|H|,|K|) = 1$, $|H \cap K| = 1$ and $H \cap K = \{1\}$. Let $h \in H$ and $k \in K$. Since both H and K are normal subgroups of G,

$$K \supseteq KK = hKh^{-1}K \ni hkh^{-1}k^{-1} \in HkHk^{-1} = HH \subseteq H.$$

Hence $hkh^{-1}k^{-1} \in H \cap K = \{1\}$. Therefore $hkh^{-1}k^{-1} = 1$ and $hk = kh$. By (a), the mapping α is a homomorphism.

Suppose $\alpha(h,k) = 1$. Then $hk = 1$ and $h = k^{-1} \in H \cap K = \{1\}$. Hence $(h,k) = (1,1) = 1_{H \times K}$ and $\mathrm{Ker}(\alpha) = \{1\}$. Therefore α is injective. ■

Sylow's Theorem

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(i) $|\mathrm{Syl}_p(G)| = |G : N_G(P)| \equiv 1 \pmod{p}$.

(ii) Let Q be a p-subgroup of G and P a Sylow p-subgroup of G, then there is $g \in G$ such that $Q \subseteq gPg^{-1}$. In particular, any two Sylow p-subgroups are conjugate in G.

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4. Let G be a group of order $90 = 2 \cdot 3^2 \cdot 5$. Let P be a Sylow 5-subgroup of G, i.e., a subgroup of G of order 5.

   (a) Show that if $P \ntrianglelefteq G$ then there is a subgroup of G of order 15.

   Solution. Suppose P is not normal in G. Then there is a conjugate gPg^{-1} of P different from P and $gPg^{-1} \in \mathrm{Syl}_5(G)$ by 1(a) and (b). Therefore $|\mathrm{Syl}_5(G)| \neq 1$. Since $|\mathrm{Syl}_5(G)| = |G : N_G(P)|$ divides $|G|$ and $|\mathrm{Syl}_5(G)| \equiv 1 \pmod{5}$, the only possible value of $|G : N_G(P)|$ is 6, and $|N_G(P)| = |G|/|G : N_G(P)| = 15$. Thus G has a subgroup of order 15. ∎

   (b) Show that G always has a cyclic subgroup H of order 15.

   Solution. First we show that G always has a subgroup of order 15. By (a), we may assume that $P \lhd G$. Let $Q \in \mathrm{Syl}_3(G)$. Then $|Q| = 3^2$. Let $1 \neq x \in Q$. Then $|x| = 3$ or $|x| = 9$. If $|x| = 9$, then $|x^3| = 3$. Hence there is always an element x of order 3. Let $R = \langle x \rangle$. Since $P \lhd G$, $PR \leq G$ by 1 (c) and $|PR| = 15$, as $|PR|$ is at most $3 \cdot 5 = 15$ and divisible by $|P| = 5$ and $|R| = 3$.

   Let H be a subgroup of order 15, and P a Sylow 5-subgroup of H and R a Sylow 3-subgroup of H. By (i), $|\mathrm{Syl}_3(H)| = |H : N_H(R)| \equiv 1 \pmod{3}$ and $|\mathrm{Syl}_5(H)| = |H : N_H(P)| \equiv 1 \pmod{5}$. Since both of them are divisors of 15, the only possibility is 1. Therefore $P \lhd H$ and $R \lhd H$. Now by 3(b), $H = PR \simeq P \times R$. Since both P and R are prime order, they are cyclic. Let $P = \langle x \rangle$ and $R = \langle y \rangle$. Then x and y commute and $|(x, y)|$, the order of $(x, y) \in P \times R$, is divisible by both 3 and 5, hence H is cyclic. ∎

   (c) Show that G is not simple, i.e., G has a nontrivial normal subgroup. (Hint: Let R be a subgroup of order 3 in H, and consider $N_G(R)$.)

   Solution. Let H be a cyclic subgroup of order 15. By (a) and (b) we may assume that H contains P. Let R be its Sylow 3-subgroup. Let Q be a Sylow 3-subgroup of G containing R. Since every group of order p^2 is abelian for every prime p, Q is abelian. Hence $Q \subseteq N_G(R) \supseteq P$, and $|N_G(R)|$ is divisible by $|Q| = 9$ and $|P| = 5$. Therefore $|N_G(R)| = 3^2 \cdot 5$ or $N_G(R) = G$. In the latter case $R \lhd G$ and as for the first case $N_G(R) \lhd G$ as every subgroup of index 2 is normal. ∎