Algebra I: Final 2007

June 21, 2007

Division: **ID#:** **Name:**

1. Let H be a subgroup of G, i.e.,

$$xy \in H \text{ and } x^{-1} \in H \text{ for all } x, y \in H.$$

(a) Show that $HH = H = H^{-1}$.

(b) Suppose that K is also a subgroup of G. Show that HK is a subgroup of G if and only if $HK = KH$.

Division: **ID#:** **Name:**

2. Let $G = \langle a \rangle$ be a cyclic group of order 15. Let $\alpha : \boldsymbol{Z} \to G$ $(n \mapsto a^n)$ be a mapping.

   (a) Show that α is a homomorphism and that $\mathrm{Ker}(\alpha) = 15\boldsymbol{Z} = \{15m \mid m \in \boldsymbol{Z}\}$.

   (b) Show that G has a subgroup H of order 5, and determine $\alpha^{-1}(H)$.

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3. Suppose $\alpha : G \times X \to X$ $((g, x) \mapsto g \cdot x)$ defines a left action of a group G on a set X. So $1 \cdot x = x$ and $g \cdot (h \cdot x) = (gh) \cdot x$ for all $g, h \in G$ and $x \in X$. Let $x_0 \in X$ and $H = \{g \in G \mid g \cdot x_0 = x_0\}$.

   (a) Show that H is a subgroup of G.

   (b) Show that $gH = g'H \Leftrightarrow g \cdot x_0 = g' \cdot x_0$ for all $g, g' \in G$.

   (c) Show that there is a bijection between G/H and $\{g \cdot x_0 \mid g \in G\}$.

Division: **ID#:** **Name:**

4. Let S_3 be the symmetric group of degree 3.

   (a) Let K be a subgroup of order 3. Show that $K = \langle(1, 2, 3)\rangle$ and $K \lhd S_3$.

   (b) Let $\alpha : G \rightarrow S_3$ be a surjective homomorphsm. Show that G has a normal subgroup N such that $|G : N| = 2$.

   (c) Suppose a finite simple group G has a subgroup H of index $|G : H| = 3$. Show that $|G| = 3$. Recall that a group $G \neq 1$ is simple if its normal subgroups are G and 1 only.

Please write your message: (1) Comments on group theory. Suggestions for improvements of this course.

(2) Comments on the educaion at ICU. Suggestions for improvements.

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1. Let H be a subgroup of G, i.e.,

$$xy \in H \text{ and } x^{-1} \in H \text{ for all } x, y \in H.$$

(a) Show that $HH = H = H^{-1}$.

Solution. Since H is a subgroup of G, $HH \subseteq H$ and $H^{-1} \subseteq H$. Since $1 \in H$, $H = 1H \subseteq HH$. Hence $HH = H$. Since $H = (H^{-1})^{-1}$, $H = (H^{-1})^{-1} \subseteq H^{-1}$. Hence $H = H^{-1}$ and $HH = H = H^{-1}$. ■

(b) Suppose that K is also a subgroup of G. Show that HK is a subgroup of G if and only if $HK = KH$.

Solution. Since $H \leq G \Leftrightarrow HH \subseteq H$ and $H^{-1} \subseteq H$, (a) implies the following. A nonempty subset H of a group G is a subgroup if and only if $H = HH = H^{-1}$. Now

$$HK \leq G \Leftrightarrow HK = HKHK = (HK)^{-1}.$$

Suppose $HK \leq G$. Then $HK = (HK)^{-1} = K^{-1}H^{-1} = KH$ as H and K are subgroups of G. Suppose $HK = KH$. Then

$$HKHK = HHKK = HK, \text{ and } (HK)^{-1} = K^{-1}H^{-1} = KH = HK.$$

Hence $HK \leq G \Leftrightarrow HK = KH$. ■

2. Let $G = \langle a \rangle$ be a cyclic group of order 15. Let $\alpha : \boldsymbol{Z} \to G$ $(n \mapsto a^n)$ be a mapping.

(a) Show that α is a homomorphism and that $\mathrm{Ker}(\alpha) = 15\boldsymbol{Z} = \{15m \mid m \in \boldsymbol{Z}\}$.

Solution. Let $n \in \mathrm{Ker}(\alpha)$. Then $1 = \alpha(n) = a^n$. Since G is a cyclic group of order 15, $15 \mid n$. Conversely, if $n = 15m$ for some integer m, then $\alpha(n) = a^n = a^{15m} = (a^{15})^m = 1$. Hence $n = 15m \in \mathrm{Ker}(\alpha)$. ■

(b) Show that G has a subgroup H of order 5, and determine $\alpha^{-1}(H)$.

Solution. Let $H = \langle a^3 \rangle = \{1, a^3, a^6, a^9, a^{12}\}$. Then $|H| = 5$ as a^i with $i = 0, 1, 2, \ldots, 14$ are all distinct. We claim that $\alpha^{-1}(H) = 3\boldsymbol{Z} = \{3m \mid m \in \boldsymbol{Z}\}$. Clearly if $n = 3m$ for some $m \in \boldsymbol{Z}$, then $\alpha(n) = a^n = a^{3m} \in H$. If $\alpha(n) \in H$, then there exists $m \in \boldsymbol{Z}$ such that $\alpha(n) = a^n = a^{3m}$. Hence $\alpha(n - 3m) = 1$ and $n - 3m \in \mathrm{Ker}(\alpha) = 15\boldsymbol{Z}$. Hence $n \in 3m + 15\boldsymbol{Z} \subseteq 3\boldsymbol{Z}$. This proves our claim. ■

3. Suppose $\alpha : G \times X \to X$ $((g, x) \mapsto g \cdot x)$ defines a left action of a group G on a set X. So $1 \cdot x = x$ and $g \cdot (h \cdot x) = (gh) \cdot x$ for all $g, h \in G$ and $x \in X$. Let $x_0 \in X$ and $H = \{g \in G \mid g \cdot x_0 = x_0\}$.

(a) Show that H is a subgroup of G.

Solution. We use the criterion in Problem 1. Let $h_1, h_2 \in H$. Then

$$(h_1h_2)\cdot x_0 = h_1\cdot(h_2\cdot x_0) = h_1\cdot x_0 = x_0,\ h_1^{-1}\cdot x_0 = h_1^{-1}\cdot(h_1\cdot x_0) = (h_1^{-1}h_1)\cdot x_0 = 1\cdot x_0 = x_0.$$

Hence $h_1h_2 \in H$ and $h_1^{-1} \in H$. ■

(b) Show that $gH = g'H \Leftrightarrow g \cdot x_0 = g' \cdot x_0$ for all $g, g' \in G$.

Solution. Suppose $gH = g'H$. Since $1 \in H$, $g \in gH = g'H$ and there exists $h \in H$ such that $g = g'h$. Now $g \cdot x_0 = (g'h) \cdot x_0 = g' \cdot (h \cdot x_0) = g' \cdot x_0$. Conversely assume $g \cdot x_0 = g' \cdot x_0$. Then

$$(g^{-1}g') \cdot x_0 = g^{-1} \cdot (g' \cdot x_0) = g^{-1}(g \cdot x_0) = (g^{-1}g) \cdot x_0 = x_0.$$

Thus $g^{-1}g' \in H$. Therefore $g' \in gH$ and $g'H = gH$. This is because

$$g'H \subseteq gHH \subseteq gH = g'g'^{-1}gH = g'(g^{-1}g')^{-1}H \subseteq g'H^{-1}H \subseteq g'H. \quad \blacksquare$$

(c) Show that there is a bijection between G/H and $\{g \cdot x_0 \mid g \in G\}$.

Solution. Let $\beta : G/H \to \{g \cdot x_0 \mid g \in G\}$ $(gH \mapsto g \cdot x_0)$. The equivalence in (b) implies that the mapping β is well-defined and injective. Moreover, by definition, the mapping is surjective. Hence β is a bijection. $\blacksquare$

4. Let S_3 be the symmetric group of degree 3.

(a) Let K be a subgroup of order 3. Show that $K = \langle (1,2,3) \rangle$ and $K \lhd S_3$.

Solution. $S_3 = \{1, (1,2), (1,3), (2,3), (1,2,3), (1,3,2)\}$. Let K be a subgroup of order 3. Let $1 \neq x \in K$. Then $\langle x \rangle$ is a subgroup of K. Hence the order of x must divide the order of K, which is 3. Hence x is of order 3. So $x = (1,2,3)$ or $(1,3,2) = (1,2,3)^2$. Hence $K = \langle (1,2,3) \rangle = \langle (1,3,2) \rangle$. Let $g \in S_3$ and $x \in K$. Then $(gxg^{-1})^3 = gx^3g^{-1} = 1$ and hence $gxg^{-1} \in K$. Hence $K \lhd S_3$. $\blacksquare$

(b) Let $\alpha : G \to S_3$ be a surjective homomorphsm. Show that G has a normal subgroup N such that $|G : N| = 2$.

Solution. Let K be a normal subgroup of S_3 in (a). Let $N = \alpha^{-1}(K)$. Let $\gamma : G \to S_3/K$ $(x \mapsto \alpha(x)K)$. (This is a composition of two surjective homomorphisms, i.e., α with a canonical homomorphism $\pi : S_3 \to S_3/K$ $(x \mapsto xK)$.) Then this is a surjective homomorphism and the kernel of γ is $\alpha^{-1}(K) = N$. Hence N is normal in G and $G/N \simeq S_3/K$. Since $|S_3/K| = |S_3|/|K| = 2$, $2 = G/N = |G : N|$. $\blacksquare$

(c) Suppose a finite simple group G has a subgroup H of index $|G : H| = 3$. Show that $|G| = 3$. Recall that a group $G \neq 1$ is simple if its normal subgroups are G and 1 only.

Solution. Let $\alpha : G \times G/H \to G/H$ $((g, xH) \mapsto gxH)$. Then α defines a left action of G on the set G/H of cardinality three. Hence there is a group homomorphism $\hat{\alpha} : G \to S_3$. Since the kernel of this homomorphism is contained in H, $\mathrm{Ker}(\hat{\alpha}) \neq G$. Since G is simple, $\mathrm{Ker}(\hat{\alpha}) = 1$ and G is isomorphic to a group of S_3 of order divisible by three. Hence $\mathrm{Im}(\hat{\alpha}) = K$ or S_3. If $\mathrm{Im}(\hat{\alpha}) = S_3$, G has a normal subgroup of index 2 by (b). This is impossible as G is simple. Hence $G \simeq \mathrm{Im}(\hat{\alpha})$ and $\mathrm{Im}(\hat{\alpha})$ is of order three. Hence it is K and is cyclic or order three. $\blacksquare$